THÉORIE DES ÉQUATIONS D'ÉVOLUTION

Jean-Yves CHEMIN
Laboratoire J.-L. Lions
Université Paris 6, Case 187
75 232 Paris Cedex 05, France
adresse électronique: chemin@ann.jussieu.fr

Chapter 1

A first approach of evolution equations

The purpose of this course is to provide some basic techniques in order to study evolution partial differential equations. In such equations, one variable (namely the time variable) plays a special role. Let us first present three examples of linear partial differential equations which we shall meet later on in the course :

- the heat equation which is a model for so called parabolic equations

$$\partial_t u - \Delta u = 0$$

- the wave wave equation, which is a model for so called parabolic equations

$$\partial_t^2 u - \Delta u = 0$$

- Schrödinger equation which is a model for so called dispersive equation

$$\partial_t u + i\Delta u = 0.$$

To each of this equation, will correspond a non linear model in the following.

The heat equation will appear in the following system which is a model for the description of the evolution of an incompressible viscous fluid. A fluid is described by a time dependant vector fields v which is supposed to describe the speed of a pointwise particle located in x at time t. The system is the following

$$\begin{cases} \partial_t v + v \cdot \nabla v - \nu \Delta v &= -\nabla p \\ \operatorname{div} v &= 0 \\ v_{|t=0} &= v_0 \end{cases}$$

Here, ν denotes a postive real number which represents the viscosity of the fluid. This system is known as the incompressible Navier-Stokes system.

For the wave equation, the following system is related to gas dynamics. The unknown is the couple (ρ, v) which satisfies

$$\left\{\begin{array}{rcl} \partial_t\rho + v\cdot\nabla\rho + \rho\,\mathrm{div}\,v & = & 0\\ \partial_t v + v\cdot\nabla v + \dfrac{1}{\rho}\nabla p & = & 0 \end{array}\right.$$

with $p = A\rho^\gamma$. Here, ρ is a scalar function with values in $\mathbb{R}^+_\star$ and represents the density of the particules of the gas at time t in the point x and v a time dependant vector field which describes the speed of a particule located in x at time t.

It will be clear later on that we have to change the unknowns defining

$$c \stackrel{\mathrm{def}}{=} \frac{2}{\gamma-1}\left(\frac{\partial p}{\partial\rho}\right)^{\frac12} = \frac{(2\gamma A)^{\frac12}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}.$$

The first equation becomes

$$\partial_t c + v\cdot\nabla c + \frac{\gamma-1}{2}c\,\mathrm{div}\,v = 0.$$

About the second one, let us observe that

$$\frac{\gamma-1}{2}c\nabla c = \frac{1}{\rho}\nabla p.$$

The Euler system related to gas dynamics becomes

$$\left\{\begin{array}{rcl} \partial_t c + v\cdot\nabla c + \dfrac{\gamma-1}{2}c\,\mathrm{div}\,v & = & 0\\ \partial_t v + v\cdot\nabla v + \dfrac{\gamma-1}{2}c\,\nabla c & = & 0. \end{array}\right. \tag{1.1}$$

Let us assume that the solution is "small", i.e. is a perturbation of magnitude ε of a stationnary flat state $v = 0$ and $c = \overline{c}$, by an easy computation of the coefficients of the powers of ε, we infer

$$\left\{\begin{array}{rcl} \partial_t c + \dfrac{\gamma-1}{2}\overline{c}\,\mathrm{div}\,v & = & 0\\ \partial_t v + \dfrac{\gamma-1}{2}\overline{c}\,\nabla c & = & 0. \end{array}\right. \tag{1.2}$$

An obvious computation ensures that

$$\partial_t^2 c - \left(\frac{\gamma-1}{2}\right)^2\overline{c}^2\Delta c = 0.$$

This equation is called "acoustic waves equation".

Then, we shall study non linear Schrödinger equations of the type

$$\partial_t u + \frac{i}{2}\Delta u = \pm|u|^{p-1}u$$

for a real number greater or equal to 1.

1.1 A review on ordinary differential equation

Before starting the study of evolution partial differential equation, let us have a look on basic properties of ordinary differential equations.

1.1.1 The linear case

Let E be a Banach space, I an open interval of $\mathbb{R}$ and A a map from I to $\mathcal{L}(E)$, the set of continuous linear maps from E into E. We want to solve the equation

$$(ODE)\left\{\begin{array}{rcl}\dot{u}\stackrel{\rm def}{=}\dfrac{du}{dt} & = & A(t)u(t)\\ u(0) & = & u_0.\end{array}\right.$$

The proof of the existence and uniqueness of solutions of this equation is very simple. Let λ be a positive real number, let us introduce the space E_λ definied by

$$E_\lambda=\Big\{u\in C(I,E)\ /\ \|u\|_\lambda\stackrel{\rm def}{=}\sup_{t\in I}\|u(t)\|\exp\Big(-\lambda\int_0^t\|A(t')\|_{\mathcal{L}(E)}dt'\Big)<\infty\Big\}.$$

The solution of (ODE) are the same as the solutions of

$$Lu=u_0\quad\text{with}\quad Lu(t)\stackrel{\rm def}{=}u(t)-\int_0^t A(t')u(t')dt'.$$

We have

$$\|(Lu-u)(t)\|\leq\int_0^t\|A(t')\|_{\mathcal{L}(E)}\|u(t')\|dt'.$$

Thus we deduce that

$$\begin{aligned}&\|(Lu-u)(t)\|\exp\Big(-\lambda\int_0^t\|A(t')\|_{\mathcal{L}(E)}dt'\Big)\\ &\leq\int_0^t\exp\Big(-\lambda\int_{t'}^t\|A(t'')\|_{\mathcal{L}(E)}dt''\Big)\|A(t')\|_{\mathcal{L}(E)}\exp\Big(-\lambda\int_0^{t'}\|A(t'')\|_{\mathcal{L}(E)}dt''\Big)\|u(t')\|dt'.\end{aligned}$$

By definition of $\|\cdot\|_\lambda$, we infer that

$$\|(Lu-u)(t)\|\exp\Big(-\lambda\int_0^t\|A(t')\|_{\mathcal{L}(E)}dt'\Big)\leq\frac{1}{\lambda}\|u\|_\lambda$$

and thus that $\|Lu-u\|_\lambda\leq\lambda^{-1}\|u\|_\lambda$. This implies that, for λ greater than 1, L is invertible in $\mathcal{L}(E)$. Then the proof of the existence and uniqueness of solutions is achieved.

1.1.2 The case of non linear equations with almost lipschitz vector field

We still work in a Banach space E and an interval I of $\mathbb{R}$. Let F be a function of $I\times E$ into E.

In the whole of this section, μ will denote a functionfrom $\mathbb{R}^+$ into itsself, vanishing at 0, positive outside 0, continuous and non decreasing.

Definition 1.1.1 *Let (X,d) and (Y,δ) be two metric spaces. We denote by $\mathcal{C}_\mu(X,Y)$ the set of the bounded functions from X into Y such that a constant C exists such that, for any $(x,y)\in X^2$, we have*

$$\delta(u(x),u(y))\leq C\mu(d(x,y)).$$

Remark If (Y,δ) is a Banach space (which we denote $(E,\|\cdot\|)$ in this case), the space $\mathcal{C}_\mu(X,E)$ is a Banach space equipped with the norm

$$\|u\|_\mu = \|u\|_{L^\infty} + \sup_{(x,y)\in X\times X,\, x\neq y} \frac{\|u(x)-u(y)\|}{\mu(d(x,y))}\cdot$$

The following theorem provides hypotheses for which there is existence and uniqueness of integral curve for an ordinary differential equation.

Theorem 1.1.1 *Let E be a Banach space, Ω an open subset of E , I an open interval of $\mathbb{R}$ and (t_0,x_0) an element of $I\times\Omega$. Let us consider a function F of $L^1_{loc}(I;\mathcal{C}_\mu(\Omega;E))$. Let us assume in addition that*

$$\int_0^1 \frac{dr}{\mu(r)} = +\infty. \tag{1.3}$$

Then an interval J exists such that $t_0\in J\subset I$ and such that the equation

$$(ODE) \quad x(t) = x_0 + \int_{t_0}^t F(t',x(t'))dt'$$

has a unique continuous solution defined on the interval J.

Remark If $\mu(r)=r$, this theorem is nothing more than the familiar Cauchy-Lipschitz theorem. But let us point out that other functions satisfy the hypotheses of the theorem; for instance the function defined by $\mu(r)=-r\log r$ for $r\leq e^{-1}$ and $\mu(r)=e^{-1}$ if not.

In order to prove, let us begin by the proof of uniqueness of trajectories. Let $x_1(t)$ and $x_2(t)$ two solutions of (ODE) defined on a neighbourhoood $\widetilde{J}$ of t_0 with this the same initial data x_0. Let us denote

$$\rho(t) \stackrel{\text{def}}{=} ||x_1(t)-x_2(t)||.$$

As F belongs to $L^1_{loc}(I;\mathcal{C}_\mu(\Omega,E))$, we have

$$0\leq\rho(t)\leq\int_{t_0}^t \gamma(t')\mu(\rho(t'))dt' \quad \text{with} \quad \gamma\in L^1_{loc}(I) \quad \text{and} \quad \gamma\geq 0. \tag{1.4}$$

In the case when $\mu(r)=r$, Gronwall lemma implies that $\rho\equiv 0$. Let us recall a version ogf Gronwalll lemma which will be useful in the following.

Lemma 1.1.1 *Let f and g be two C^0 (resp. C^1) non negative functions on $[t_0,T]$. Let $\mathcal{A}$ be a continuous functions on $[t_0,T]$. Suppose that, for t in $[t_0,T]$,*

$$\frac{1}{2}\frac{d}{dt}\, g^2(t) \leq \mathcal{A}(t)\, g^2(t) + f(t)g(t). \tag{1.5}$$

Then for any time t in $[t_0,T]$, we have

$$g(t) \leq g(t_0)\exp\int_{t_0}^t \mathcal{A}(t')\,dt' + \int_{t_0}^t f(t')\exp\Big(\int_{t'}^t \mathcal{A}(t'')\,dt''\Big)\,dt'\,.$$

Let us define

$$g_{\mathcal{A}}(t) \stackrel{\text{def}}{=} g(t)\exp\Big(-\int_{t_0}^t \mathcal{A}(t')\,dt'\Big) \quad \text{and} \quad f_{\mathcal{A}}(t) \stackrel{\text{def}}{=} f(t)\exp\Big(-\int_{t_0}^t \mathcal{A}(t')\,dt'\Big).$$

Obviously, we have $\frac{1}{2}\frac{d}{dt}g_{\mathcal{A}}^2 \leq f_{\mathcal{A}}g_{\mathcal{A}}$ so that for any positive ε,

$$\frac{d}{dt}(g_{\mathcal{A}}^2+\varepsilon^2)^{\frac{1}{2}} \leq \frac{g_{\mathcal{A}}}{(g_{\mathcal{A}}^2+\varepsilon^2)^{\frac{1}{2}}} f_{\mathcal{A}} \leq f_{\mathcal{A}}.$$

By integration, we get

$$(g_{\mathcal{A}}^2(t)+\varepsilon^2)^{\frac{1}{2}} \leq (g_{\mathcal{A}}^2(0)+\varepsilon^2)^{\frac{1}{2}} + \int_0^t f_{\mathcal{A}}(t')\,dt'.$$

Having ε tend to 0 gives the result.

The key lemma for the proof of Theorem 1.1.1 is the following.

Lemma 1.1.2 *Let ρ be a measurable non negative function, γ a non negative function locally integrable and μ a continuous non decreasing function. Let us assume that, for a non negative real number a, the function ρ satisfies*

$$\rho(t) \leq a + \int_{t_0}^t \gamma(t')\mu(\rho(t'))dt'. \tag{1.6}$$

If a is positive, then we have

$$-\mathcal{M}(\rho(t)) + \mathcal{M}(a) \leq \int_{t_0}^t \gamma(t')dt' \quad \text{with} \quad \mathcal{M}(x) = \int_x^1 \frac{dr}{\mu(r)}. \tag{1.7}$$

If $a = 0$ and if μ satisfies (1.3), then the function ρ is identically 0.

In order to prove this lemma, let us assume that a is positive and let us define

$$R_a(t) \stackrel{\text{def}}{=} a + \int_{t_0}^t \gamma(t')\mu(\rho(t'))dt'.$$

The function R_a is a continuous non decreasing function. Thus, we have,

$$\dot{R}_a(t) = \gamma(t)\mu(\rho(t)).$$

As the function μ is non decreasing, we have

$$\dot{R}_a(t) \leq \gamma(t)\mu(R_a(t)). \tag{1.8}$$

The function R_a is positive. As the function $\mathcal{M}$ is C^1 on $]0,\infty[$, Inequality (1.8) implies that

$$-\frac{d}{dt}\mathcal{M}(R_a(t)) = \frac{\dot{R}_a(t)}{\mu(R_a(t))} \leq \gamma(t).$$

By inegration, we get (1.7) using that the function $-\mathcal{M}$ is increasing and that $\rho \leq R_a$.

Let us assume now that $a = 0$ and let us proceed by contraposition. Let us assume that the function ρ is not identically 0 near t_0. As the function μ is non decreasing, we can substitute $\sup_{t'\in[t_0,t]} \rho(t')$ to the function ρ (we continue to use the same notation ρ). A real number $t_1 > t_0$ exists, such that $\rho(t_1) > 0$. As the function ρ satisfies (1.6) for $a = 0$,it satisfies also this inequality for any positive a'. It comes from (1.7) that

$$\forall a' > 0\,,\ \mathcal{M}(a') \leq \int_{t_0}^{t_1} \gamma(t')dt' + \mathcal{M}(\rho(t_1)).$$

This implies that the integral

$$\int_0^1 \frac{dr}{\mu(r)}$$

is convergent; the proof of the lemma is done.

Thanks to Inequality (1.4), the uniqueness of integral curve issued from a point is an obvious consequence of Lemma 1.1.2.

Let us prove the existence. In order to do so, let us consider the classsical Picard scheme

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} F(t', x_k(t'))dt'.$$

Let us skip the proof of the fact that, for a small enough interval J, the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(J)$. Let us prove that this sequence is a Cauchy one in the space of continuous functions from J to E. In order to do so, let us define

$$\rho_{k+1,n}(t) \stackrel{\text{def}}{=} ||x_{k+1+n}(t) - x_{k+1}(t)||.$$

It turns out that

$$0 \leq \rho_{k+1,n}(t) \leq \int_{t_0}^{t} \gamma(t')\mu(\rho_{k,n}(t'))dt'$$

Let us define $\rho_k(t) \stackrel{\text{def}}{=} \sup_n ||x_{k+1+n}(t) - x_{k+1}(t)||$. As the function μ is non decreasing, we have

$$0 \leq \rho_{k+1}(t) \leq \int_{t_0}^{t} \gamma(t')\mu(\rho_k(t'))dt'.$$

Thanks to Fatou lemma, we get, using that the function μ is non decreasing,

$$\widetilde{\rho}(t) \stackrel{\text{def}}{=} \limsup_{k\to+\infty} \rho_k(t) \leq \int_{t_0}^{t} \gamma(t')\mu(\widetilde{\rho}(t'))dt'.$$

Applying again Lemma 1.1.2, we find that $\widetilde{\rho}(t)$ is identically 0 near t_0; this concludes the proof of Theorem 1.1.1.

Let us point out that the concepts of iterative scheme and of Cauchy sequence plays a key role.

1.1.3 Blow up criteria

The existence and uniqueness theorem for ordinary differential equations is a local theorem. Let us investigate what can be necessary conditions for a blow up phenomena.

Proposition 1.1.1 *Let F be a function of $\mathbb{R}\times E$ in E satisfying the hypothesis of Theorem 1.1.1 in any point x_0 of E. Let us assume in addition that a locally bounded function M from $\mathbb{R}^+$ into $\mathbb{R}^+$ and a locally integrable function β from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that*

$$\|F(t,u)\| \leq \beta(t)M(\|u\|).$$

then, if the maximal interval of definition is $]T_\star, T^\star[$*, then, if* $T^\star$ *is finite,*

$$\limsup_{t\to T^\star} \|u(t)\| = \infty.$$

Let us first prove that, if we consider a time $T > T_0$ such that $\|u(t)\|$ is bounded on the interval $[T_0, T[$, then we can define the solution on a larger interval $[T_0, T_1]$ with $T_1 > T$. As the function u is bounded on the interval $[T_0, T[$, the hypothesis on F that, for any t of the interval $[T_0, T[$, we have

$$\|F(t, u(t))\| \leq C\beta(t).$$

The function β being integrable on the interval $[T_0, T]$, we have deduce que, for any ε strictement positive, it exists a positive real number η such that, pour tout t and t' such that $T-t < \eta$ and $T - t' < \eta$,

$$\|u(t) - u(t')\| < \varepsilon.$$

The space E being complete, an element $u_\star$ of E exists such that

$$\lim_{t\to T^\star} u(t) = u_\star.$$

Applying Theorem 1.1.1, we construct solution of (ODE) on some $[T_+^\star, T_1]$ and the continuous function defined by induction on the interval $[T_0, T_1]$ is a solution of the equation (ODE) on the interval $[T_0, T_1]$.

Corollary 1.1.1 *Under the hypothesis of Proposition 1.1.1, if we have in addition that*

$$\|F(t, u)\| \leq M\|u\|^2,$$

then, if the interval $]T_\star, T^\star[$ *is the maximal interval of definition of* u *and if* $T^\star$ *is finite, then*

$$\int_{t_0}^{T^\star} \|x(t)\|dt = \infty.$$

The solution satisfies, for any $t \geq t_0$

$$\|x(t)\| \leq \|x(t_0)\| + M\int_{t_0}^{t} \|x(t')\|^2 dt'. \tag{1.9}$$

Gronwall's Lemma implies that

$$\|x(t)\| \leq \|x_0\| \exp\Big(M\int_0^t \|x(t')\|dt'\Big).$$

A more precise way of proving this result is the following.

Let $T \stackrel{\text{def}}{=} \sup\{t \in [t_0, T^\star[\,/\ \|x(t)\| \leq 2\|x(t_0)\|\}$. For any $t \in [t_0, T^\star[$, we have, using (1.9),

$$\|x(t)\| \leq \|x(t_0)\| + 4M(t - t_0)\|x(t_0)\|^2.$$

Thus we infer

$$\forall t \in \Big[t_0, \min\Big\{T, t_0 + \frac{1}{4M\|x(t_0\|)}\Big\}\Big[, \quad \|x(t)\| \leq 2\|x_0\|.$$

Thanks to Proposition 1.1.1, we have

$$T^\star - t_0 \geq \frac{c}{\|x_0\|}\cdot$$

Applying again this result at time $t \in [t_0, T^\star[$, we find that

$$\forall t \in [t_0, T^\star[\,, \quad \|x(t)\| \geq \frac{c}{T^\star - t}\cdot$$

Exercice 1.1.1 *Let F a function defined on $\mathbb{R} \times E$ such that*

$$\sup_{x\in E} \|F(t,x)\| + \sup_{\substack{(x,y)\in E^2\\ 0<\|x-y\|\leq e^{-1}}} \frac{\|F(t,x)-F(t,y)\|}{-\|x-y\|\log\|x-y\|} \leq \beta(t) \quad \textit{with} \quad \beta \in L^1_{loc}(\mathbb{R}).$$

1) Prove that it existsd a map ψ from $\mathbb{R} \times E$ into E such that

$$\psi(t,x) = x + \int_0^t F(t, \psi(t,x))dt$$

2) Prove that, for any t, $\psi(t,\cdot)$ defines a homeomorphism of E such that

$$|x-y| \leq e^{-\exp\int_0^t \beta(s)ds} \Rightarrow |\psi(t,x) - \psi(t,y)| \leq |x-y|^{\exp - \int_0^t \beta(s)ds} e^{-\exp - \int_0^t \beta(s)ds}.$$

1.1.4 A compactness theorem : Peano's theorem

The theorem is the following.

Theorem 1.1.2 (Peano) *Let I be an open interval of $\mathbb{R}$. Let us consider a function f from $I \times \mathbb{R}^d$ into $\mathbb{R}^d$ such that*

- *For any compact K of $\mathbb{R}^d$, the function $t \mapsto \|f(t)\|_{L^\infty(K)}$ is locally integrable,*
- *For any t of I, the function $x \mapsto f(t,x)$ is continuous on $\mathbb{R}^d$.*

Then, for any point (t_0, x_0) of $I \times \mathbb{R}^d$, an open interval $J \subset I$ containing t_0 and a continuous function x on J exists such that

$$(ODE) \qquad x(t) = x_0 + \int_{t_0}^t f(t', x(t'))dt'.$$

The structure of the proof is at least as intesting as the result itself. This proof will be a model for the proof of existence of weak solutions for ther incompressible Navier-Stokes equation we shall study in Chater 4.

There are three steps in the proof

- we regularize the function f and we apply Cauchy-Lipschitz's Theorem to the sequence of regularized functions; Proposition 1.1.1 ensures that the solutions of the regularized problem have a commun interval of definition,
- then, we prove that the sequence of those solutions of the regularized problem are relatively compact in the space $C(J, \mathbb{R}^d)$,
- as a conclusion, we pass to the limit.

Let us procede to a classical regularization; let χ a non negative function of $\mathcal{D}(B(0,1))$ the integral of which is 1. Let us define $\chi_n(x) \stackrel{\rm def}{=} n^d\chi(nx)$ and $f_n(t) = \chi_n \star f(t)$. We have

$$\|f_n(t)\|_{L^\infty(K)} \leq \|f(t)\|_{L^\infty(K+B(0,n^{-1}))}.$$

Morevoer, we have

$$\|\partial_j f_n(t)\|_{L^\infty(K)} \leq C(n+1)\|f(t)\|_{L^\infty(K+B(0,n^{-1}))}.$$

We can apply Cauchy-Lipschitz's Theorem of to the function f_n. Let J_n the maximal interval of definition of x_n. Let J an interval ouvert such that

$$\int_J \|f(t)\|_{L^\infty(B(x_0,2)}dt \leq 1.$$

Let us define $t_n \stackrel{\rm def}{=} \sup\Big\{t \in [t_0,\infty[\cap J \cap J_n \,/\, \forall t' \leq t\,,\ x(t') \in B(x_0,1)\Big\}$. For any $t \leq t_n$, we have

$$\begin{aligned} \|x_n(t) - x_0\| &\leq \int_J \|f_n(t)\|_{L^\infty(B(x_0,1))}dt \\ &\leq \int_J \|f(t)\|_{L^\infty(B(x_0,2))}dt \\ &\leq 1. \end{aligned}$$

Thus $t_n \geq \sup J \cap J_n$. working in the same way for the times less to t_0, we find, using Proposition 1.1.1 that, for any n, $J \subset J_n$. This concludes the first part of the proof.

We have

$$\forall t \in J\,,\ X(t) \stackrel{\rm déf}{=} \{x_n(t),\ n \in \mathbb{N}\} \subset B(x_0,1).$$

As we work on a finite dimensionnal space, $X(t)$ is relatively compact. Moreover, we have

$$\begin{aligned} \|x_n(t) - x_n(t')\| &\leq \Big|\int_t^{t'} \|f_n(t'')\|_{L^\infty(B(x_0,1))}dt''\Big| \\ &\leq \Big|\int_t^{t'} \|f(t'')\|_{L^\infty(B(x_0,2))}dt''\Big|. \end{aligned}$$

Thus, for any positive ϵ, it exists a positive real number α such that

$$\forall (t,t') \in J^2\,,\ |t-t'| < \alpha \Longrightarrow \|x_n(t) - x_n(t')\| < \epsilon.$$

In other words, the family $(x_n)_{n\in\mathbb{N}}$ is equicontinuous on J. Ascoli's Theorem ensures that the set of functions x_n is relatively compact in $C(J;\mathbb{R}^d)$. Thus we can extract a subsequence which converge uniformely on J to a function x of $C(J;\mathbb{R}^d)$. Let omit to note the extraction in the following.

Now let us pass to the limit. For any t of J; we have

$$\|f_n(t,x_n(t) - f(t,x(t))\| \leq \|f_n(t) - f(t)\|_{L^\infty(B(x_0,1))} + \|f(t,x_n(t)) - f(t,x(t))\|.$$

Thus for any t of J, we have

$$\lim_{n\to\infty} f_n(t,x_n(t)) = f(t,x(t)).$$

Moreover, $\|f_n(t,x_n(t)\| \leq \|f(t)\|_{L^\infty(B(x_0,2))}$. Lebesgue's Theorem ensures that, for any t, we have

$$\lim_{n\to\infty} \int_{t_0}^t f_n(t',x_n(t'))dt' = \int_{t_0}^t f(t',x(t'))dt'.$$

The theorem is proved.

Remarks

- All the theorems and all the proofs of this chapter must be known.
- To know more about ordinary differential equations and their historical aspect of Osgood's theory, see the book by T.M. Fleet, *Differential analysis*, Cambridge University Press, 1980.
- To know more about non lipschiztian vector fields satisfying Osgood condition, see the book by J.-Y. Chemin, *Fluides parfaits incompressibles*, Astérisque, **230**, 1995 or its english version *Incompressible perfect fluids*, Oxford University Press, 1998.

Chapter 2

Sobolev spaces

Introduction

In this course, we shall restrict ourselves to Sobolev spaces modeled on L^2. These spaces definitely play a crucial role in the study of partial differential equations, linear or not. The key tool will be the Fourier transform.

2.1 Definition of Sobolev spaces on $\mathbb{R}^d$

Definition 2.1.1 *Let s be a real number, a tempered distribution u belongs to the Sobolev space of index s, denoted $H^s(\mathbb{R}^d)$, or simply H^s if no confusion is possible, if and only if*

$$\widehat{u} \in L^2_{loc}(\mathbb{R}^d) \quad \text{and} \quad \widehat{u}(\xi) \in L^2(\mathbb{R}^d; (1+|\xi|^2)^s d\xi).$$

and we note

$$\|u\|^2_{H^s} = \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi.$$

Proposition 2.1.1 *For any s real number, the space H^s, equipped with the norm $\|\cdot\|_{H^s}$, is a Hilbert space.*

The fact that the norm $\|\cdot\|_{H^s}$ comes from the scalar product

$$(u|v)_{H^s} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} (1+|\xi|^2)^s \widehat{u}(\xi)\overline{\widehat{v}(\xi)} d\xi$$

is obvious. Let us prove that this space is complete. Let $(u_n)_{n\in\mathbb{N}}$ a Cauchy sequence of H^s. By definition of the norm, the sequence $(\widehat{u}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of the space $L^2(\mathbb{R}^d; (1+|\xi|^2)^s d\xi)$. Thus, a function $\widetilde{u}$ exists in the space $L^2(\mathbb{R}^d; (1+|\xi|^2)^s d\xi)$ such that

$$\lim_{n\to\infty} \|\widehat{u}_n - \widetilde{u}\|_{L^2(\mathbb{R}^d;(1+|\xi|^2)^s d\xi)} = 0. \tag{2.1}$$

In particular, the sequence$(\widehat{u}_n)_{n\in\mathbb{N}}$ tends to $\widetilde{u}$ in the space $\mathcal{S}'$ of tempered distribtions. Let $u = \mathcal{F}^{-1}\widetilde{u}$. As the Fourier transform is an isomorphism of $\mathcal{S}'$, the sequence $(u_n)_{n\in\mathbb{N}}$ tends to u in the space $\mathcal{S}'$, and also in H^s thanks to (2.1).

Shortly said, this is nothing more than observing that the Fourier transform is an isometric isomorphism from H^s onto $L^2(\mathbb{R}^d; (1+|\xi|^2)^s d\xi)$.

Proposition 2.1.2 *Let s be a non negative integer, the space $H^s(\mathbb{R}^d)$ is the space of functions u of L^2 all the derivatives of which of order less or equal to m are distributions which belongs to L^2. Moreover, the space H^m equipped with the norm*

$$\widetilde{\|}u\|_{H^m}^2 \stackrel{\text{def}}{=} \sum_{|\alpha|\leq m} \|\partial^\alpha u\|_{L^2}^2$$

is a Hilbert space and this norm is equivalent to the norme $\|\cdot\|_{H^s}$.

The fact that

$$\widetilde{\|}u\|_{H^m}^2 = \widetilde{(}u|u)_{H^m} \quad \text{with} \quad \widetilde{(}u|v)_{H^m} \stackrel{\text{déf}}{=} \sum_{|\alpha|\leq m} \int_{\mathbb{R}^d} \partial^\alpha u(x)\overline{\partial^\alpha v(x)}dx.$$

ensures that the norm $\|\cdot\|_{H^m}$ comes from a scalar product. Moreover, a constant C exists such that

$$\forall \xi \in \mathbb{R}^d\,,\ C^{-1}\Big(1 + \sum_{0<|\alpha|\leq m} |\xi|^{2|\alpha|}\Big) \leq (1+|\xi|^2)^s \leq C\Big(1 + \sum_{0<|\alpha|\leq m} |\xi|^{2|\alpha|}\Big). \tag{2.2}$$

As the Fourier transform is, up to a constant, an isometric isomorphism from L^2 onto L^2, we have

$$\partial^\alpha u \in L^2 \Longleftrightarrow \xi^\alpha \widehat{u} \in L^2.$$

Thus, we have deduce that

$$u \in H^m \Longleftrightarrow \forall \alpha \,/\, |\alpha| \leq m\,,\ \partial^\alpha u \in L^2.$$

Inequality (2.2) ensures the equivalence of the two norms using again the fact that the Fourier transform is a isometric isomorphism up to a constant. The proposition is proved.

Exercice 2.1.1 *Prove that the space $\mathcal{S}$ is continuously included in the space H^s for any real s.*

Exercice 2.1.2 *Prove that the mass of Dirac δ_0 belongs to the space $H^{-\frac{d}{2}-\varepsilon}$ for any positive real number ε. Prove that δ_0 does not belong to the space $H^{-\frac{d}{2}}$.*

Exercice 2.1.3 *Prove that, for any distribution to support compact u, it exists a real number s such that u belongs to the Sobolev space H^s.*

Exercice 2.1.4 *Prove that the constant 1 does not belong to H^s for any real number s.*

Proposition 2.1.3 *Let s a real number of the interval $]0,1[$. Prove that the space H^s is the space des functions u of L^2 such that*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}} dxdy.$$

Moreover, a constant C exists such that, for any function u of H^s, we have

$$C^{-1}\|u\|_{H^s}^2 \leq \|u\|_{L^2}^2 + \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}} dxdy \leq C\|u\|_{H^s}^2.$$

Thanks to Fourier-Plancherel identity,we can write that

$$\int_{\mathbb{R}^d} \frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}}dx = \int_{\mathbb{R}^d} \frac{|e^{i(y|\xi)}-1|^2}{|y|^{d+2s}}|\widehat{u}(\xi)|^2 d\xi < \infty.$$

It turns out that

$$\begin{aligned} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}}dxdy &= \int_{\mathbb{R}^d} F(\xi)|\widehat{u}(\xi)|^2 d\xi \quad \text{with} \\ F(\xi) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \frac{|e^{i(y|\xi)}-1|^2}{|y|^{2s}}\frac{dy}{|y|^d}. \end{aligned}$$

By an obvious change of variable, we see that the function F is radial and homogeneous of degree $2s$. Thus

$$F(\xi) = |\xi|^{2s}\int \frac{|e^{iy_1}-1|^2}{|y|^{2s}}\frac{dy}{|y|^d}.$$

Let us prove now an interpolation inequality which will be very useful.

Proposition 2.1.4 *If $s=\theta s_1+(1-\theta)s_2$ with $\theta\in[0,1]$, then, we have*

$$\|u\|_{H^s} \leq \|u\|_{H^{s_1}}^{\theta}\|u\|_{H^{s_2}}^{1-\theta}.$$

The proof consits in applying Hölder inequality with the measure $|\widehat{u}(\xi)|^2 d\xi$ and the two functions $(1+|\xi|^2)^{\theta s_1}$ and $(1+|\xi|^2)^{(1-\theta)s_2}$.

Theorem 2.1.1 *Let s a real quelconque;*

- *the space $\mathcal{D}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$,*
- *the multiplication by a function of $\mathcal{S}$ est a continuous function of H^sinto lui-même.*

In order to prove the first point of this theorem, let us consider a distribution u of H^s such that, for any test function φ, we have $(\varphi|u)_{H^s}=0$. This means that, for any test function φ, we have

$$\int_{\mathbb{R}^d} \widehat{\varphi}(\xi)(1+|\xi|^2)^s\overline{\widehat{u}(\xi)}d\xi = 0.$$

As $\mathcal{S}$ is continuously included in H^s, as $\mathcal{D}$ is dense in $\mathcal{S}$, and as the Fourier transform an isomorphism of $\mathcal{S}$, we have, for any function f of $\mathcal{S}$,

$$\int_{\mathbb{R}^d} f(\xi)(1+|\xi|^2)^s\overline{\widehat{u}(\xi)}d\xi = 0.$$

As $\mathcal{S}$ is dense in L^2, this implies that $(1+|\xi|^2)^s\widehat{u}(\xi)=0$, thus $\widehat{u}=0$ and thus $u=0$.

Let us prove now the second second point of the theorem. This proof is presented here just for culture. We know that

$$\widehat{\varphi u} = (2\pi)^{-d}\widehat{\varphi}\star\widehat{u}.$$

The point is to estimate the L^2 norm of the function defined by

$$U(\xi) = (1+|\xi^2|)^{\frac{s}{2}}\int_{\mathbb{R}^d} |\widehat{\varphi}(\xi-\eta)|\times|\widehat{u}(\eta)|d\eta.$$

Let us define $I_1(\xi) = \{\eta \,/\, 2|\xi-\eta| \leq |\eta|\}$ and $I_2(\xi) = \{\eta \,/\, 2|\xi-\eta| \geq |\eta|\}$. It is clear that we have

$$\begin{aligned} U(\xi) &= U_1(\xi) + U_2(\xi) \quad \text{with} \\ U_j(\xi) &= (1+|\xi|^2)^{\frac{s}{2}} \int_{I_j(\xi)} |\widehat{\varphi}(\xi-\eta)| \times |\widehat{u}(\eta)| d\eta. \end{aligned}$$

Let us first observe that, if $\eta \in I_1(\xi)$, then

$$\frac{1}{2}|\eta| \leq |\xi| \leq \frac{3}{2}|\eta|.$$

We deduce that, for any real number s, a constant C exists such that, for any couple (ξ, η) such that η belongs to $I_1(\xi)$,

$$(1+|\xi|^2)^s \leq C(1+|\eta|^2)^s.$$

Thus it turns out that

$$U_1(\xi) \leq C \int_{\mathbb{R}^d} |\widehat{\varphi}(\xi-\eta)|(1+|\eta|^2)^{\frac{s}{2}} |\widehat{u}(\eta)| d\eta.$$

As $\widehat{\varphi}$ belongs to $\mathcal{S}$, it also belongs to L^1 and we have

$$\|U_1\|_{L^2} \leq C \|\widehat{\varphi}\|_{L^1} \|u\|_{H^s}.$$

But if η belongs to $I_2(\xi)$, we have

$$\frac{1}{2}|\xi-\eta| \leq |\xi| \leq \frac{3}{2}|\xi-\eta| \quad \text{and} \quad |\eta| \leq \frac{5}{2}|\xi-\eta|.$$

Then we deduce that

$$\begin{aligned} U_2(\xi) &\leq C(1+|\xi|^2)^{\frac{|s|}{2}} \int_{\mathbb{R}^d} |\widehat{\varphi}(\xi-\eta)|(1+|\eta|^2)^{\frac{|s|}{2}} (1+|\eta|^2)^{\frac{s}{2}} |\widehat{u}(\eta)| d\eta \\ &\leq C \int_{\mathbb{R}^d} |\widehat{\varphi}(\xi-\eta)|(1+|\xi-\eta|^2)^{|s|} (1+|\eta|^2)^{\frac{s}{2}} |\widehat{u}(\eta)| d\eta. \end{aligned}$$

We know that $\widehat{\varphi}$ belongs to $\mathcal{S}$. A constant C exists such that

$$|\widehat{\varphi}(\zeta)| \leq C(1+|\zeta|^2)^{-\frac{d+1}{2}-|s|}.$$

Thus it tuns out that

$$U(\xi) \leq C \int_{\mathbb{R}^d} (1+|\xi-\eta|^2)^{-\frac{d+1}{2}} (1+|\eta|^2)^{\frac{s}{2}} |\widehat{u}(\eta)| d\eta.$$

Thus $\|U_2\|_{L^2} \leq C\|u\|_{H^s}$; this concludes the proof of the theorem.

Exercice 2.1.5 *Let $\mathcal{F}L^1 = \{u \in \mathcal{S}' \,/\, \widehat{u} \in L^1\}$. Prove that, for any non negative real number s, the product is a bilinear continuous map from $\mathcal{F}L^1 \cap H^s \times \mathcal{F}L^1 \cap H^s$ into $\mathcal{F}L^1 \cap H^s$. What happens when s is greater than $d/2$?*

Exercice 2.1.6 *Let s a real number greater than $1/2$. Prove that the map γ defined by*

$$\gamma \left\{ \begin{array}{ccc} \mathcal{D}(\mathbb{R}^d) & \longrightarrow & \mathcal{D}(\mathbb{R}^{d-1}) \\ \varphi & \longmapsto & \gamma(\varphi) : (x_2, \cdots, x_d) \mapsto \varphi(0, x_2, \cdots, x_d) \end{array} \right.$$

can be extended in a continuous onto map from $H^s(\mathbb{R}^d)$ onto $H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$.
Hint : *Write*

$$\mathcal{F}_{\mathbb{R}^{d-1}}\varphi(0, \xi_2, \cdots, \xi_d) = (2\pi)^{-1} \int_{\mathbb{R}} \widehat{\varphi}(\xi_1, \xi_2, \cdots, \xi_d) d\xi_1.$$

and for the fact that the map in onto, observe that, if

$$u = (2\pi)^{-(n-1)} C_s \mathcal{F}^{-1} \left(\frac{(1 + |\xi'|^2)^{s-\frac{1}{2}}}{(1 + |\xi|^2)^s} \widehat{v}(\xi') \right),$$

then $u \in H^s$ and $\gamma(u) = v$.

Let us prove a theorem which describes the dual of the space H^s.

Theorem 2.1.2 *The bilinear form B defined by*

$$B \left\{ \begin{array}{ccc} \mathcal{S} \times \mathcal{S} & \rightarrow & \mathbb{C} \\ (u, \varphi) & \mapsto & \displaystyle\int_{\mathbb{R}^d} u(x)\varphi(x)dx \end{array} \right.$$

can b extended as a bilinear form continuous from $H^{-s} \times H^s$ to $\mathbb{C}$. Moreover, the map δ_B defined by

$$\delta_B \left\{ \begin{array}{ccc} H^{-s} & \longrightarrow & (H^s)' \\ u & \longmapsto & \delta_B(u) : (\varphi) \mapsto B(u, \varphi) \end{array} \right.$$

is a linear and isometric isomorphism (up to a constant), which means that the bilinear form B identifies the space H^{-s} to the dual space of H^s.

The important point of the proof of this theorem is inverse Fourier formula which ensures that, for any couple (u, φ) of functions of $\mathcal{S}$, we have

$$\begin{aligned} B(u, \varphi) &= \int_{\mathbb{R}^d} u(x)\varphi(x)dx \\ &= \int_{\mathbb{R}^d} u(x)\mathcal{F}(\mathcal{F}^{-1}\varphi)(x)dx \\ &= \int_{\mathbb{R}^d} \widehat{u}(\xi)(\mathcal{F}^{-1}\varphi)(\xi)d\xi \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{u}(\xi)\widehat{\varphi}(-\xi)d\xi. \end{aligned} \tag{2.3}$$

Multiplying and dividing by $(1 + |\xi|^2)^{\frac{s}{2}}$, we immediately get thanks to Cauchy-Schwarz inequality,

$$|B(u, \varphi)| \leq (2\pi)^{-d} \|u\|_{H^s} \|\varphi\|_{H^{-s}}.$$

Thus the first point of the theorem. The fact that the map δ_B is one to one comes from the fact that if, for any function $\varphi \in \mathcal{S}$, we have $B(u, \varphi) = 0$, then u is 0. We have to prove that the map is onto. In fact, we shall prove that δ_B is one to one and onto. For any real number σ, the Fourier transform is an isometric (up to a constant) isomorphism from H^σ onto $L^2(\mathbb{R}^d,(1+|\xi|^2)^\sigma d\xi)$. Let us now consider the bilinear form $\check{B}$ defined by

$$\check{B}\left\{\begin{array}{ccc} L^2(\mathbb{R}^d,(1+|\xi|^2)^{-s}d\xi)\times L^2(\mathbb{R}^d,(1+|\xi|^2)^{s}d\xi) & \longrightarrow & \mathbb{C} \\ (\phi,f) & \longmapsto & (2\pi)^{-d}\displaystyle\int_{\mathbb{R}^d} f(\xi)\phi(-\xi)d\xi. \end{array}\right.$$

If we prove that

$$\delta_B = {}^t\mathcal{F}\delta_{\check{B}}\mathcal{F}, \tag{2.4}$$

then Theorem 2.1.2 will be proved. Indeed, as $\mathcal{F}$ is an isomorphism from H^s onto $L^2(\mathbb{R}^d,(1+|\xi|^2)^s d\xi)$, the map ${}^t\mathcal{F}$ is an isomorphism from $(L^2(\mathbb{R}^d,(1+|\xi|^2)^s d\xi))'$ onto $(H^s)'$. We know that $\delta_{\check{B}}$ is an isomorphism from the space $(L^2(\mathbb{R}^d,(1+|\xi|^2)^s d\xi))'$ onto the space $L^2(\mathbb{R}^d,(1+|\xi|^2)^{-s}d\xi)$.

In order to prove Formula (2.4), let us write that

$$\begin{aligned}\langle {}^t\mathcal{F}\delta_{\check{B}}\mathcal{F}u,\varphi\rangle &= \langle \delta_{\check{B}}\mathcal{F}u,\mathcal{F}\varphi\rangle \\ &= \delta_{\check{B}}(\mathcal{F}u,\mathcal{F}\varphi).\end{aligned}$$

Thanks to Identity (2.3), we have $\langle {}^t\mathcal{F}\delta_{\check{B}}\mathcal{F}u,\varphi\rangle = \langle \delta_B(u),\varphi\rangle$. Thus the theorem is proved.

2.2 Sobolev embeddings

The purpose of this section is the study of embedding properties of Sobolev spaces $H^s(\mathbb{R}^d)$ into L^p spaces . Let us prove the following theorem.

Theorem 2.2.1 *If s is greater than $d/2$, then the space H^s is continuously included in the space of continuous functions which tend to 0 at infinity. If s is a positive real number less than $d/2$, then the space H^s is continuously included in $L^{\frac{2d}{d-2s}}$ and we have*

$$\|f\|_{L^p}\leq C\|f\|_{\dot H^s}\quad \textit{with}\quad \|f\|_{\dot H^s}\stackrel{\text{def}}{=}\Big(\int_{\mathbb{R}^d}|\xi|^{2s}|\widehat{f}(\xi)|^2 d\xi\Big)^{\frac12}.$$

The first point of this theorem is very easy to prove. Let us use the fact that

$$\|u\|_{L^\infty}\leq (2\pi)^{-d}\|\widehat{u}\|_{L^1} \tag{2.5}$$

Indeed, if s is greater than $d/2$, we have,

$$|\widehat{u}(\xi)|\leq (1+|\xi|^2)^{-s/2}(1+|\xi|^2)^{s/2}|\widehat{u}(\xi)|. \tag{2.6}$$

The fact that s is greater than $d/2$ implies that the function

$$\xi\mapsto (1+|\xi|^2)^{-s/2}$$

belongs to L^2. Thus, we have

$$\|\widehat{u}\|_{L^1}\leq\Big(\int(1+|\xi|^2)^{-s}d\xi\Big)^{\frac12}\|u\|_{H^s}.$$

The first point of the theorem is proved.

The proof of the second point is more delicate. A way to unterstand the index $p = 2d/(d-2s)$ is the use of a scaling argument. Let us consider a function v on $\mathbb{R}^d$ and let us denote by v_λ the function $v_\lambda(x) = v(\lambda x)$. We have

$$\|v_\lambda\|_{L^p} = \lambda^{-\frac{d}{p}}\|v\|_{L^p}$$

and also

$$\begin{aligned} \int |\xi|^{2s}|\widehat{v_\lambda}(\xi)|^2 d\xi &= \lambda^{-2d}\int |\xi|^{2s}|\widehat{v}(\lambda^{-1}\xi)|^2 d\xi \\ &= \lambda^{-d+2s}\|v\|^2_{\dot{H}^s}, \end{aligned}$$

with

$$\|v\|^2_{\dot{H}^s} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^{2s}|\widehat{v}(\xi)|^2 d\xi.$$

The two quantities $\|\cdot\|_{L^p}$ and $\|\cdot\|_{\dot{H}^s}$ have the same scaling, which means that they have the same behaviour with respect to changes of unit. Thus, it make sense to compare them.

Multiplying f by a positive real number, it is enough to prove the inequality in the case when $\|f\|_{\dot{H}^s} = 1$. On utilise then the fact that for any p de the interval $]1, +\infty[$, we have, for any function measurable f,

$$\|f\|^p_{L^p} = p\int_0^\infty \lambda^{p-1} m(|f| > \lambda) d\lambda.$$

Let us decompose f in a low and in a high frequencies by writing

$$f = f_{1,A} + f_{2,A} \quad \text{with} \quad f_{1,A} = \mathcal{F}^{-1}(\mathbf{1}_{B(0,A)}\widehat{f}) \quad \text{and} \quad f_{2,A} = \mathcal{F}^{-1}(\mathbf{1}_{B^c(0,A)}\widehat{f}). \tag{2.7}$$

As the support of the Fourier transform of $f_{1,A}$ is compact, the function $f_{1,A}$ is bounded and more precisely,

$$\begin{aligned} \|f_{1,A}\|_{L^\infty} &\leq (2\pi)^{-d}\|\widehat{f_{1,A}}\|_{L^1} \\ &\leq (2\pi)^{-d}\int_{B(0,A)} |\xi|^{-s}|\xi|^s|\widehat{f}(\xi)|d\xi \\ &\leq (2\pi)^{-d}\left(\int_{B(0,A)} |\xi|^{-2s}d\xi\right)^{\frac{1}{2}} \\ &\leq \frac{C}{(d-2s)^{\frac{1}{2}}}A^{\frac{d}{2}-s}. \end{aligned} \tag{2.8}$$

The triangle inequality implies that, for any positive real number A,

$$(|f| > \lambda) \subset (2|f_{1,A}| > \lambda) \cup (2|f_{2,A}| > \lambda).$$

Using Inequality (2.8), we have

$$A = A_\lambda \stackrel{\text{def}}{=} \left(\frac{\lambda(d-2s)^{\frac{1}{2}}}{4C}\right)^{\frac{p}{d}} \Longrightarrow m\left(|f_{1,A}| > \frac{\lambda}{2}\right) = 0.$$

Thus we deduce that

$$\|f\|^p_{L^p} = p\int_0^\infty \lambda^{p-1} m(2|f_{2,A_\lambda}| > \lambda) d\lambda.$$

it is well known (this is Bienaimé-Tchebychev inequality) that

$$
\begin{aligned}
m\left(|f_{2,A_\lambda}| > \frac{\lambda}{2}\right) &= \int_{\left(|f_{2,A_\lambda}|>\frac{\lambda}{2}\right)} dx \\
&\leq \int_{\left(|f_{2,A_\lambda}|>\frac{\lambda}{2}\right)} \frac{4|f_{2,A_\lambda}(x)|^2}{\lambda^2} dx \\
&\leq 4\frac{\|f_{2,A_\lambda}\|^2_{L^2}}{\lambda^2}.
\end{aligned}
$$

For such a choice of A, we have

$$
\|f\|^p_{L^p} \leq 4p\int_0^\infty \lambda^{p-3}\|f_{2,A_\lambda}\|^2_{L^2}d\lambda. \tag{2.9}
$$

As the Fourier transform is (up to a constant) an isometric isomorphism of L^2, we have

$$
\|f_{2,A_\lambda}\|^2_{L^2} = (2\pi)^{-d}\int_{(|\xi|\geq A_\lambda)} |\widehat{f}(\xi)|^2 d\xi.
$$

Thanks to Inequality (2.9), we get

$$
\|f\|^p_{L^p} \leq 4p(2\pi)^{-d}\int_{\mathbb{R}_+\times\mathbb{R}^d} \lambda^{p-3}\mathbf{1}_{\{(\lambda,\xi)\,/\ |\xi|\geq A_\lambda\}}(\lambda,\xi)|\widehat{f}(\xi)|^2 d\xi d\lambda.
$$

By definition of A_λ, we have

$$
|\xi| \geq A_\lambda \Longleftrightarrow \lambda \leq C_\xi \stackrel{\text{déf}}{=} \frac{4C}{(d-2s)^{\frac{1}{2}}}|\xi|^{\frac{d}{p}}.
$$

Fubini's theorem implies that

$$
\begin{aligned}
\|f\|^p_{L^p} &\leq 4p(2\pi)^{-d}\int_{\mathbb{R}^d}\left(\int_0^{C_\xi}\lambda^{p-3}d\lambda\right)|\widehat{f}(\xi)|^2 d\xi \\
&\leq 4\frac{p(2\pi)^d}{p-2}\left(\frac{4C}{(d-2s)^{\frac{1}{2}}}\right)^{p-2}\int_{\mathbb{R}^d}|\xi|^{\frac{d(p-2)}{p}}|\widehat{f}(\xi)|^2 d\xi.
\end{aligned}
$$

As $2s = \dfrac{d(p-2)}{p}$, the theorem is proved.

Corollary 2.2.1 *Let $p\in]2,\infty[$, and $s > s_p \stackrel{\text{def}}{=} d\Big(\frac{1}{2}-\frac{1}{p}\Big)$. We have*

$$
\|u\|_{L^p} \leq C\|u\|^{1-\theta}_{L^2}\|u\|^{\theta}_{\dot{H}^s} \quad \textit{with} \quad \theta = \frac{s_p}{s}\,\cdot
$$

The proof of this corollary is an application of the above theorem together with the fact that

$$
\|u\|_{\dot{H}^{\theta s_1+(1-\theta)s_2}} \leq \|u\|^{\theta}_{\dot{H}^s_1}\|u\|^{1-\theta}_{\dot{H}^{s_2}}.
$$

2.3 Homogeneous Sobolev spaces

Definition 2.3.1 *Let s be a real number, the homogeneous Sobolev space $\dot H^s$ is the space of tempered distribtions such that $\widehat u$ belongs to L^1_{loc} and satisfies*

$$\|u\|^2_{\dot H^s} \stackrel{\rm def}{=} \int_{\mathbb{R}^d} |\xi|^{2s}|\widehat u(\xi)|^2 d\xi < \infty.$$

These spaces (or at least theirs norms) naturally appeared in the proof of Theorem 2.2.1. The $\|\cdot\|_{\dot H^s}$ norm has the following scaling property

$$\|f(\lambda\cdot)\|_{\dot H^s} = \lambda^{-\frac{d}{2}+s}\|f\|_{\dot H^s}.$$

These spaces are different from the inhomogeneous H^s spaces. Let us notice that if s is positive, then H^s is included in $\dot H^s$ but that if s is negative, then $\dot H^s$ is included in H^s. The inhomongeneous spaces is a decreasing family of spaces (with respect to the index s). The homogeneous ones are not comparable together.

We shall only consider theses homogeneous spaces in the case when s is less than the half dimension.

Proposition 2.3.1 *If $s < d/2$, then the space $\dot H^s$ is a Banach space.*

Let $(u_n)_{n\in\mathbb{N}}$ a Cauchy sequence of $\dot H^s$. The sequence $(\widehat u_n)_{n\in\mathbb{N}}$ is a Cauchy one in the Banach space $L^2(\mathbb{R}^d\setminus\{0\};|\xi|^{2s}d\xi)$. Let f be its limit. It is clear that f belongs to $L^1_{loc}(\mathbb{R}^d\setminus\{0\})$. Moreover,

$$\int_{B(0,1)} |f(\xi)|d\xi \leq \Big(\int_{\mathbb{R}^d} |\xi|^{2s} f(\xi)|^2 d\xi\Big)^{\frac12}\Big(\int_{B(0,1)} |\xi|^{-2s}d\xi\Big)^{\frac12} < \infty$$

because s is less than the half-dimension. Thus $\widehat f$ belongs to $\mathcal{S}'$ and to L^1_{loc}. Thus $u \stackrel{\rm déf}{=} \mathcal{F}^{-1}f$ is well defined, belongs to $\dot H^s$, and is the limit of the sequence $(u_n)_{n\in\mathbb{N}}$ in the sense of the norm $\dot H^s$.

Exercice 2.3.1 *1) Prove that the space*

$$\mathcal{B} \stackrel{\rm def}{=} \{u \in \mathcal{S}'(\mathbb{R}^d)\,,\ \widehat u \in L^1(B(0,1);d\xi)\cap L^2(\mathbb{R}^d;|\xi|^{2s}d\xi)\}$$

equipped with the norme $N(u) \stackrel{\rm def}{=} \|\widehat u\|_{L^1(B(0,1))} + \|u\|_{\dot H^s}$ is a Banach space.

2) Let $s \geq d/2$. Give an example of a sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{B}$, bounded in $\dot H^s(\mathbb{R}^d)$, such that

$$\lim_{n\to\infty} N(f_n) = +\infty.$$

3) Then deduce that $(\dot H^s, \|\cdot\|_{\dot H^s})$ is not a Banach space.

Exercice 2.3.2 *Prove that, if $k\in\mathbb{N}$, then we have*

$$\dot H^{-k}(\mathbb{R}^d) = \Big\{u\in\mathcal{S}'(\mathbb{R}^d)\,,\ u = \sum_{|\alpha|=k}\partial^\alpha f_\alpha \quad with \quad f_\alpha\in L^2\Big\}.$$

Prove that a constant C exists such that

$$C^{-1}\|u\|_{\dot H^{-k}} \leq \inf\Big\{\Big(\sum_{|\alpha|=k}\|f_\alpha\|^2_{L^2}\Big)^{\frac12} /\ u = \sum_{|\alpha|=k}\partial^\alpha f_\alpha\Big\} \leq C\|u\|_{\dot H^{-k}}.$$

2.4 The spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$

Definition 2.4.1 *Let Ω a domain of $\mathbb{R}^d$, the space $H_0^1(\Omega)$ is defined as the closure of $\mathcal{D}(\Omega)$ in the sense of the norm $H^1(\mathbb{R}^d)$.*

The space $H^{-1}(\Omega)$ is the set of distributions u on Ω such that

$$\|u\|_{H^{-1}(\Omega)} \stackrel{def}{=} \sup_{\substack{f\in\mathcal{D}(\Omega)\\ \|f\|_{H_0^1(\Omega)}\leq 1}} |\langle u, f\rangle| < \infty.$$

Proposition 2.4.1 *The space $H_0^1(\Omega)$ is a Hilbert space equipped with the norm*

$$\left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

The proof is an easy exercice left to the reader. The space $H^{-1}(\Omega)$ can be indentified to the dual space of $H_0^1(\Omega)$ thanks to the following theorem.

Theorem 2.4.1 *The bilinear map defined by*

$$B\left\{\begin{array}{ccc} H^{-1}(\Omega)\times\mathcal{D}(\Omega) & \longrightarrow & \mathbb{C}\\ (u,\varphi) & \longmapsto & \langle u,\varphi\rangle\end{array}\right.$$

can be extended to a bilinear continuous map from $H^{-1}(\Omega)\times H_0^1(\Omega)$ into $\mathbb{C}$, still denoted by B. Moreover, the map δ_B defined by

$$\delta_B\left\{\begin{array}{ccc} H^{-1}(\Omega) & \longrightarrow & (H_0^1(\Omega))'\\ u & \longmapsto & \delta_B(u)(\varphi)\stackrel{def}{=} B(u,\varphi)\end{array}\right.$$

is a linear isometric isomorphism between the space $H^{-1}(\Omega)$ and the dual space of $H_0^1(\Omega)$.

The fact that the bilinear map B can be extended because B is uniformely continuous. Let ℓ a linear form continuous on $H_0^1(\Omega)$. Its restriction on $\mathcal{D}(\Omega)$ is a distribution u on Ω such that

$$\forall\varphi\in\mathcal{D}(\Omega)\,,\ \langle u,\varphi\rangle = \langle \ell,\varphi\rangle.$$

By definition of the norm on $(H_0^1(\Omega))'$, the theorem is proved.

Theorem 2.4.2 (Poincaré Inequality) *Let Ω be* **bounded** *open subset of $\mathbb{R}^d$. A constant C exists such that*

$$\forall\varphi\in H_0^1(\Omega)\,,\ \|\varphi\|_{L^2}\leq C\left(\sum_{j=1}^d\|\partial_j\varphi\|^2_{L^2}\right)^{\frac{1}{2}}.$$

Let R a positive real number such that Ω is included in $]-R,R[\times\mathbb{R}^{d-1}$.Then, for any test function φ, we have

$$\varphi(x_1,\cdots,x_d) = \int_{-R}^{x_1}\frac{\partial\varphi}{\partial y_1}(y_1,x_2,\cdots,x_d)dy_1.$$

Cauchy-Schwarz Inequality implies that

$$|\varphi(x_1,\cdots,x_d)|^2 \leq 2R\int_{-R}^{x_1}\left|\frac{\partial\varphi}{\partial y_1}(y_1,x_2,\cdots,x_d)\right|^2 dy_1.$$

By integration in x_1, we get

$$\int_\Omega |\varphi(x_1,\cdots,x_d)|^2 dx_1 \leq 2R\int_{\Omega\times]-R,R[}\left|\frac{\partial\varphi}{\partial y_1}(y_1,x_2,\cdots,x_d)\right|^2 dy_1.$$

Then, integrating with respect to the other $d-1$ variables, we find

$$\begin{aligned}\int_\Omega |\varphi(x_1,\cdots,x_d)|^2 dx &\leq 2R\int_{\Omega\times]-R,R[}\left|\frac{\partial\varphi}{\partial y_1}(y_1,x_2,\cdots,x_d)\right|^2 dy_1 dx_2\cdots dx_d\\ &\leq 4R^2\sum_{j=1}^d \|\partial_j\varphi\|_{L^2}^2.\end{aligned}$$

As $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, the theorem is proved. It obviously implies the following corollary.

Corollary 2.4.1 *The space $H_0^1(\Omega)$ equipped with the norm*

$$u \longmapsto \left(\sum_{j=1}^d \|\partial_j u\|_{L^2}^2\right)^{\frac{1}{2}} \stackrel{\text{def}}{=} \|\nabla u\|_{L^2}$$

is a Hilbert space and the this norm is equivalent to the previous one.

In order to conclude this chapter, let us prove the following very important compactness theorem.

Theorem 2.4.3 *Let K be a compact of $\mathbb{R}^d$ and (s,s') a couple of real numbers such that $s'<s$. Let us denote by H_K^s the space of distributions of $H^s(\mathbb{R}^d)$ the support of which is included in K. The embedding of H_K^s in $H_K^{s'}$ is compact.*

Before proving this theorem, let us give some immediate corollaries.

Theorem 2.4.4 *Let Ω be a bounded open subset of $\mathbb{R}^d$ with $d\geq 2$. If p is a real number less than*

$$p_c \stackrel{\text{def}}{=} \frac{2d}{d-2},$$

then the embedding of $H_0^1(\Omega)$ in $L^p(\Omega)$ is compact.

Let us prove Theorem 2.4.3. Without any loss on generalitry, we can assume that the compact K is included in the interior of the cube $[0,2\pi]^d$. Now let us introduce the linear map defined by

$$P\left\{\begin{array}{ccc}\mathcal{D}_K & \longrightarrow & C^\infty(\mathbb{T}^d)\\ u & \longmapsto & \displaystyle\sum_{k\in 2\pi\,\mathbb{Z}^d} u(x-k)\end{array}\right.$$

where $\mathcal{D}_K$ denotes the set of smooth compactly supported functions the support of which is included in K. This map can be extended to a linear continuous map of H_K^s in $H^s(\mathbb{T}^d)$ defined by

$$H^s(\mathbb{T}^d) = \{u\in\mathcal{D}(\mathbb{T}^d)\ /\ \|u\|_{H^s(\mathbb{T}^d)}^2 \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}}(1+|n|^2)^s|\widehat{u}(n)|^2 < +\infty\}.$$

This comes from the fact that a constant C exists such that

$$\forall \varphi \in \mathcal{D}_K \,, \ \|P\varphi\|_{H^s(\mathbb{T}^d)} \leq C \|\varphi\|_{H^s}. \tag{2.10}$$

In order to prove this, let us consider a function χ of $\mathcal{D}]0, 2\pi[^d)$ with value 1 near K. Then, let us write

$$\begin{aligned} \widehat{\varphi}(n) &= \int \varphi(x) e^{-inx} dx \\ &= \int \varphi(x)\chi(x) e^{-inx} dx \\ &= (2\pi)^{-d} \int \widehat{\varphi}(\xi) \mathcal{F}(\chi e^{-inx})(\xi) d\xi \\ &= (2\pi)^{-d} \int \widehat{\varphi}(\xi) \widehat{\chi}(n-\xi) d\xi. \end{aligned}$$

As χ is a smooth compactly supported function, for any integer N, a constant C_N exists such that

$$|\widehat{\varphi}(n)| \leq C_N \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(\xi)|}{(1+|n-\xi|)^{-N}} d\xi.$$

The result is an obvious consequence of the following two lemmas.

Lemma 2.4.1 *For any $(a,b) \in \mathbb{R}^d$, for any $s \in \mathbb{R}$, we have*

$$(1+|a+b|^2)^{\frac{s}{2}} \leq 2^{\frac{|s|}{2}} (1+|a|^2)^{\frac{|s|}{2}} (1+|b|^2)^{\frac{s}{2}}.$$

Lemma 2.4.2 *Let (X_{j,μ_j}) two measured spaces and k a function measurable from $X_1 \times X_2$ into $\mathbb{R}$ such that*

$$M \stackrel{\text{déf}}{=} \max\Big\{ \sup_{x_2 \in \mathcal{X}_2} \int_{X_1} |k(x_1,x_2)| d\mu_1(x_1), \ \sup_{x_1 \in \mathcal{X}_1} \int_{X_2} |k(x_1,x_2)| d\mu_2(x_2) \Big\} < \infty.$$

Then the map defined by

$$(Kf)(x_2) = \int_{X_1} k(x_1,x_2) f(x_1) d\mu_1(x_1),$$

maps $L^p(X_1, d\mu_1)$ into $L^p(X_2, d\mu_2)$ for any $p \in [1,\infty]$. More precisely, we have

$$\|Kf\|_{L^p(X_2,d\mu_2)} \leq M \|f\|_{L^p(X_1,d\mu_1)}.$$

Proof of Lemma 2.4.1 Let us first observe that

$$1+|a+b|^2 \leq 1+2(|a|^2+|b|^2) \leq 2(1+|a|^2)(1+|b|^2).$$

Taking the power $s/2$ of this inequality, we find the result for non negative $s \geq 0$. In the case when s is negative, we have

$$(1+|b|^2)^{-\frac{s}{2}} \leq 2^{-\frac{s}{2}} (1+|a+b|^2)^{-\frac{s}{2}} (1+|a|^2)^{-\frac{s}{2}}.$$

Thus the result is proved.

Proof of the Lemma 2.4.2 Let g be an element of norm 1 of $L^{p'}(X_2, d\mu_2)$. We have

$$\int_{X_2} |Kf(x_2)|\, |g(x_2)| d\mu_2(x_2) \leq \int_{X_1\times X_2} |k(x_1,x_2)|\, |f(x_1)|\, |g(x_2)| d\mu_1(x_1) d\mu_2(x_2).$$

Hölder Inequality for the measure $|k(x_1,x_2)|d\mu_1(x_1)d\mu_2(x_2)$ implies that

$$\int_{X_2} |Kf(x_2)|\, |g(x_2)| d\mu_2(x_2) \leq \Big(\int_{X_1\times X_2} |f(x_1)|^p |k(x_1,x_2)| d\mu_1(x_1) d\mu_2(x_2)\Big)^{\frac{1}{p}}$$
$$\times \Big(\int_{X_1\times X_2} |g(x_2)|^{p'} |k(x_1,x_2)| d\mu_1(x_1) d\mu_2(x_2)\Big)^{\frac{1}{p'}}.$$

Then Fubini's Theorem ensures the result.

Now let us prove that the embedding of $H^s(\mathbb{T}^d)$ in $H^{s'}(\mathbb{T}^d)$ is compact. In order to do so, let us observe that, if we define

$$i_N(\varphi) \stackrel{\text{def}}{=} (2\pi)^{-d} \sum_{n\leq |N|} \widehat{\varphi}(n) e^{i(n|x)},$$

where N denotes any integer, then we have

$$\begin{aligned}
|\varphi - i_N(\varphi)|^2_{H^{s'}(\mathbb{T}^d)} &= \sum_{n>|N|} (1+|n|^2)^{s'} |\widehat{\varphi}(n)|^2 \\
&= (1+|N|^2)^{s'-s} \sum_{n>|N|} (1+|n|^2)^{s} |\widehat{\varphi}(n)|^2 \\
&= (1+|N|^2)^{s'-s} |\varphi|^2_{H^{s'}(\mathbb{T}^d)}.
\end{aligned}$$

Thus, the embedding i of $H^s(\mathbb{T}^d)$ in $H^{s'}(\mathbb{T}^d)$ is compact as a limit of finite rank operators.

Let us consider the map defined by

$$M_\chi \left\{ \begin{array}{ccc} H^{s'}(\mathbb{T}^d) & \longrightarrow & H^{s'}(\mathbb{R}^d) \\ u & \longmapsto & \chi u \end{array} \right.$$

Using the Fourier transform of u, we get

$$\begin{aligned}
\mathcal{F}(\chi u)(\xi) &= \int_{\mathbb{R}^d} e^{-i(x|\xi)} \chi(x) u(x) dx \\
&= \sum_{n\in\mathbb{Z}^d} \widehat{u}(n) \int_{\mathbb{R}^d} e^{-i(x|\xi-n)} \chi(x) dx \\
&= \sum_{n\in\mathbb{Z}^d} \widehat{u}(n) \widehat{\chi}(n-\xi).
\end{aligned}$$

The proof of the continuity of M_χ is strictly analogous to the proof of the continuity of P,. Moreover, it is clear that the embedding of H^s_K in $H^{s'}_K$ is equal to $P\circ i\circ M_\chi$; the theorem is proved.

Remarks

The proofs of this chapter must be known except the proof of the second point of Theorem 2.1.1.

To know more about Sobolev spaces, the reader can conslut the classical book

R. A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, Vol. 65, Academic Press, 1975.

Chapter 3

Extrema problem and the least action principle

3.1 The problem of Dirichlet vu comme a problem d'extremum

In this section and also in the following one, Ω denotes a bounded domain of $\mathbb{R}^d$. Let f be an element of $H^{-1}(\Omega)$, let us consider the functionnal F defined par

$$F \left\{ \begin{array}{ccc} H^1_0(\Omega) & \longrightarrow & \mathbb{R} \\ u & \longmapsto & \frac{1}{2}\|\nabla u\|^2_{L^2} - \langle f, u\rangle. \end{array} \right.$$

Dirichlet Theorem is the following:

Theorem 3.1.1 *The functionnal F has a unique minimum which is the unique solution in $H^1_0(\Omega)$ of $-\Delta u = f$ in the distribution sense in Ω.*

In order to prove this theorem, let us observe that the functionnal F bounded from below because

$$\begin{aligned} F(u) &\geq \frac{1}{2}\|\nabla u\|^2_{L^2} - \|\nabla u\|_{L^2}\|f\|_{H^{-1}(\Omega)} \\ &\geq \frac{1}{2}\big(\|\nabla u\|_{L^2} - \|f\|_{H^{-1}(\Omega)}\big)^2 - \frac{1}{2}\|f\|^2_{H^{-1}(\Omega)}. \end{aligned} \tag{3.1}$$

The functionnal F has a lower bound m. Let us consider a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ i.e. a sequence $(u_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty} F(u_n) = m$. Using lnequality (3.1), we have

$$\|\nabla u_n\|_{L^2} \leq \big(2F(u_n) + \|f\|_{H^{-1}(\Omega)}\big)^{\frac{1}{2}} + \|f\|_{H^{-1}(\Omega)}.$$

The sequence $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of the space $H^1_0(\Omega)$ which is complete. Thus it exists a function u in $H^1_0(\Omega)$ and a subsequence of $(u_n)_{n\in\mathbb{N}}$ (which we still denote by $(u_n)_{n\in\mathbb{N}}$) such that $(u_n)_{n\in\mathbb{N}}$ tends weakly to u. Moreover, we know that the sequence $(\|\nabla u_n\|_{L^2})_{n\in\mathbb{N}}$ converges to $m + \langle f, u\rangle$. Thanks to the properties of the weak limit we have

$$\lim_{n\to\infty} \|\nabla u_n\|_{L^2} \geq \|\nabla u\|_{L^2}.$$

Let us assume that $\nabla u\|_{L^2} < \lim\limits_{n\to\infty} \|\nabla u_n\|_{L^2}$. Then, we have $F(u) < m$ which is in contradiction with the fact that m is the infimum of F. Thus

$$\lim_{n\to\infty} \|\nabla u_n\|_{L^2} = \|\nabla u\|_{L^2}$$

and then the lower bound is a minimum and the sequence $(u_n)_{n\in\mathbb{N}}$ converges strongly to u in $H^1_0(\Omega)$).

Now let us prove that u is a solution of Laplace Equation. The functionnal F is the sum of the quadratic functionnal (the norm to the square) and of of a linear functionnal (both continuous). We have, for any function h of $H^1_0(\Omega)$,

$$F(u+h) = F(u) + 2(\nabla u|\nabla h)_{L^2} - \langle f, h\rangle + \|\nabla h\|^2_{L^2}. \tag{3.2}$$

If u is a minumum, then the differential vanishes at u and thus u is a solution of Laplace Equation. Moreover, Relation (3.2) implies that the minimum is unique and it is characterised by the fact that, for any h in $H^1_0(\Omega)$, we have $(\nabla u|\nabla h)_{L^2} - \langle f, h\rangle = 0$. Thus the theorem is proved.

Exercice 3.1.1 *Let Ω a bounded domain of $\mathbb{R}^d$ and f a distribution of $H^{-1}(\Omega)$. Prove that a vector field v exists in $L^2(\Omega)$ such that* $\operatorname{div} v = f$.

Let us prove now a result about the spectral strucutre of the Laplacian in a bounded domain.

Theorem 3.1.2 *It exists a non decreasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of positive real numbers which thends to infinity and a hilbertian basis of $L^2(\Omega)$ denoted by $(e_k)_{k\in\mathbb{N}}$ such that the sequence $(\lambda_k^{-1}e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $H^1_0(\Omega)$ such that*

$$-\Delta e_k = \lambda_k e_k.$$

Moreover, if f belongs to $H^{-1}(\Omega)$, then

$$\|f\|^2_{H^{-1}(\Omega)} = \sum_k \lambda_k^{-2}(\langle f, e_k\rangle)^2.$$

Remark Thus, the space $H^{-1}(\Omega)$ is a Hilbert space and $(\lambda_k e_k)_{k\in\mathbb{N}}$ is a hilbertian basis of $H^{-1}(\Omega)$.

Proof of Theorem 3.1.2

As the space L^2 is continuously included in $H^{-1}(\Omega)$, we can define an operator B as follows:

$$B\left\{\begin{array}{lll} L^2 & \longrightarrow & H^1_0(\Omega)\subset L^2(\Omega)\\ f & \longmapsto & u\end{array}\right.$$

such that u is the solution in $H^1_0(\Omega)$ of $-\Delta u = f$. The operator B is of course continuous from $L^2(\Omega)$ into $H^1_0(\Omega)$. Thanks to Theorem 2.4.3, the operator B is compact from $L^2(\Omega)$ into $L^2(\Omega)$. Moreover, the operator B is selfadjoint and positive, i.e. that, for any couple of functions of $L^2(\Omega)$ (f, g), we have

$$(Bf|g)_{L^2} = (f|Bg)_{L^2} \quad \text{and} \quad (Bf|f)_{L^2} > 0 \quad \text{if} \quad f\neq 0.$$

By definition of B, it exists a couple of functions in $H^1_0(\Omega)$ (u, v) such that we have,

$$(Bf|g)_{L^2} = -(Bf|\Delta Bg)_{L^2} = (\nabla Bf|\nabla Bg)_{L^2}.$$

Thus the operator B is compact, selfadjoint and positive. The spectral theorem applied to B implies the existence of a non increasing sequence $(\mu_k)_{k\in\mathbb{N}}$ of positive real numbers which tends to 0 and a hilbertian basis of $L^2(\Omega)$ denoted $(e_k)_{k\in\mathbb{N}}$ such that , for any k, the function e_k belongs to $L^2(\Omega)$ and such that $Be_k = \mu_k e_k$. This implies that $-\Delta e_k = \mu_k^{-1} e_k$. We have,

$$\|f\|_{H^{-1}(\Omega)} = \sup_{(c_k)\in B_f} \langle f, \sum_k \lambda_k^{-1} c_k e_k\rangle$$

where B_f denotes the set of sequences having only a finite number of non zero terms and of ℓ^2 norm less or equal to to 1. Thus

$$\|f\|_{H^{-1}(\Omega)} = \sup_{(c_k)\in B_f} \sum_k \lambda_k^{-1}\langle f, e_k\rangle c_k = \|(\lambda_k^{-1}\langle f, e_k\rangle)_{k\in\mathbb{N}}\|_{\ell^2(\mathbb{N})}.$$

Theorem 3.1.2 is proved.

3.2 The problem of Stokes

This problem is analogous to the Dirichlet problem, but we work on the set of divergence free vector field. Nevertheless, the fact that we impose a constrain (even a linear one) of the space on which we search the minimum will introduce an important change. The Laplace eqaution will become the Stokes equation. Let us first define of the space we are going to work with.

Definition 3.2.1 *Let us denote by $\mathcal{V}_\sigma(\Omega)$ the set of divergence free vector fields whose componants are in $H_0^1(\Omega)$ and by $\mathcal{H}(\Omega)$ the closure in $(L^2(\Omega))^d$ de $\mathcal{V}_\sigma(\Omega)$.*

Let us state the analogous of Dirichlet theorem in this framework. As in the preceeding section, let us consider a vector feld f whose componants are in $H^{-1}(\Omega)$; then we define the functionnal F

$$F\left\{\begin{array}{ccc} \mathcal{V}_\sigma(\Omega) & \longrightarrow & \mathbb{R} \\ u & \longmapsto & \dfrac{1}{2}\|\nabla u\|_{L^2}^2 - \langle f, u\rangle. \end{array}\right.$$

Theorem 3.2.1 *Let $f \in \mathcal{V}'(\Omega)$. It exists a unique minimum of the functionnal F which is also the unique solution of following equation*

$$-\Delta u - f \in (\mathcal{V}_\sigma(\Omega))^\circ$$

which means that, for any vector field v of $\mathcal{V}_\sigma(\Omega)$, we have

$$-\langle \Delta u, v\rangle = \langle f, v\rangle. \tag{3.3}$$

The existence and the uniqueness of a minimum u for the functionnal F can be proved following exactly the same lines as in the case of Dirichlet problem. The fact that the differential of F vanishes at point u implies the relation (3.3).

Remarks

- The fact that a vector field g of $H^{-1}(\Omega)$ belongs the polar set (in the sense of the duality) of $H_0^1(\Omega)$ implies in particular that, for any function φ of $\mathcal{D}(\Omega)$, we have

  $$\langle g^i, -\partial_j\varphi\rangle + \langle g^j, \partial_i\varphi\rangle = 0$$

  which implies that $\partial_j g^i - \partial_i g^j = 0$, i.e. the curl of g is identically 0.

- Very simple domains exist sucht that it exists a vector field of $H^{-1}(\Omega)$ which are of divergence and of curl identically 0 and which are not gradients of functions.

Let us consider the domain of the plan $\Omega \stackrel{\text{def}}{=} \{x \in \mathbb{R}^2 \ / \ 0 < R_1 < |x| < R_2\}$ and the vector field f defined by $(-\partial_2 \log|x|, \partial_1 \log|x|)$. We have the following lemma.

Proposition 3.2.1 *The vector field f is of divergence free 0, but it is not the gradient of a function.*

The fact that f is of divergence free is obvious. The fact that its curl is 0 follows from the fact that that the function $x \mapsto \log|x|$ is harmonic on Ω. Let us assume that f is a gradient of some distribution $-p$. As f is smooth, p is also smooth. Let us consider the flow of $f = -\nabla p$. By definition of f, its trajectories are periodic. Let us consider a trajectory γ from of a point of Ω such that $f \neq 0$ (here all points are like this). We have

$$\frac{d}{dt}(p \circ \gamma)(t) = \Big(\frac{d\gamma}{dt}\Big|\nabla p(\gamma(t))\Big)_{L^2} = -|\nabla p(\gamma(t))|^2 \leq 0.$$

The fact that the derivative en $t = 0$ is négative is contradictoire with la périodicité of the trajectoire γ. La proposition 3.2.1 is provede.

As shown by the following proposition, belonging to the polar space (in the sens of the duality $H^{-1}, H_0^1(\Omega)$) of $(\mathcal{V}_\sigma(\Omega))^\circ$ is stronger than being curl free. Let us admit the following proposition .

Proposition 3.2.2 *Let f in $\mathcal{V}'(\Omega)$. If f belongs à $(\mathcal{V}_\sigma(\Omega))^\circ$ i.e. if*

$$\forall v \in \mathcal{V}_\sigma(\Omega)\,, \ \sum_{j=1}^{d} \langle f^j, v^j \rangle = 0,$$

then it exists p in $\mathcal{D}'(\Omega)$ such that $f = -\nabla p$. If the boundary of Ω is a C^1, hypersurface, then $p \in L^2(\Omega)$.

Asinto the cas of Dirichlet problem, nous allons appliquer un result of theory spectrale on les operators audoadjoints compacts pour obtenir the following theorem.

Theorem 3.2.2 *Il exists a non decreasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of reals strictement positives tendant to l'infini and a hilbertian basis of $\mathcal{H}(\Omega)$ denoted $(e_k)_{k\in\mathbb{N}}$ such that the sequence $(\lambda_k^{-1} e_k)_{k\in\mathbb{N}}$ soit une base hilbertienne of $\mathcal{V}_\sigma(\Omega)$ and such that*

$$-\Delta e_k - \lambda_k^2 e_k \in (\mathcal{V}_\sigma(\Omega))^\circ.$$

Moreover, if $f \in \mathcal{V}'(\Omega)$, alors

$$\|f\|^2_{\mathcal{V}'_\sigma(\Omega)} = \sum_{k\in N} \lambda_k^{-2} (\langle f, e_k \rangle)^2 \quad \textit{and} \quad \lim_{n\to\infty} \Big\| f - \sum_{k=0}^{n} \langle f, e_k \rangle e_k \Big\|_{(\mathcal{V}_\sigma(\Omega))'} = 0.$$

The proof is very close to the proof of Theorem 3.1.2. As the space $\mathcal{H}(\Omega)$ is continuously included in $H^{-1}(\Omega)$, we can define the operator B

$$B \left\{ \begin{array}{ccc} \mathcal{H}(\Omega) & \longrightarrow & \mathcal{V}_\sigma(\Omega) \subset \mathcal{H}(\Omega) \\ f & \longmapsto & u \end{array} \right.$$

such that u is the solution in $H_0^1(\Omega)$ of $-\Delta u - f \in (\mathcal{V}_\sigma(\Omega))^\circ$. The following of the proof is strictement analogous to the one of Dirichlet problem.

The orthogonal projection of $L^2(\Omega)$ on $\mathcal{H}(\Omega)$, denoted by $\mathbb{P}$, is the Leray projection.

3.3 How to modelize fluids using the least action principle

In this section, we shall always assume that the fluid extends to the whole space $\mathbb{R}^d$, which means that we neglect boundary effects. We want to describe the evolution of a perfect fluid between two times t_0 and t_1. A particule of fluid located at point x to l'time t_0 sera located at point $\psi_1(x)$ at the time t_1. The possible incompressibility of the fluid will be described by the fact that the map ψ_1, assumed to be a diffeomorphism, will preserved the measure, i.e. its jacobian is of determinant 1.

Let us precise the functionnal spaces we are going to work with. In all this section, we consider a diffeomorphism ψ_1 which preserves the volume if the fluid is assumed incompressible.

Definition 3.3.1 *Let us denote by $\mathcal{L}$ the space of C^1 functions from $[t_0, t_1] \times \mathbb{R}^d$ in $\mathbb{R}^d$ such that $\psi(t_0) = \mathrm{Id}$ and $\psi(t_1) = \psi_1$, such that, each time t, the function $\psi(t)$ is a diffeomorphism of $\mathbb{R}^d$ and then such that $\partial_t \psi(t)$ is continuous from $[t_0, t_1]$ into L^2.*

Let us denote by $\mathcal{L}_0$ the space des functions of $\mathcal{L}$ such that, for any time t, the diffeomorphism $\psi(t)$ preserves the measure.

A possible evolution of a compressible fluid between the situation at time t_0 and the one at time t_1 by the diffeomorphism ψ_1, is modelized by a function ψ of the space $\mathcal{L}$. A possible evolution of an incompressible fluid between the situation at time t_0 and the situation at time t_1 described by the diffeomorphism ψ_1, is modelized by a function ψ of the space $\mathcal{L}_0$. This is the point of view is called the lagragian one.

Let us defined a functionnal the extremal points of which will decribes the evolutions of the fluid.

Definition 3.3.2 *Let F a C^∞ function and ρ_0 a C^∞ positive function; Let us define the action by the map $\mathcal{A}$ defined from $\mathcal{L}$ into $\mathbb{R}$*

$$
\begin{aligned}
\mathcal{A}(\psi) &\stackrel{\text{def}}{=} \mathcal{A}_1(\psi) + \mathcal{A}_2(\psi) \quad \textit{with} \\
\mathcal{A}_1(\psi) &\stackrel{\text{def}}{=} \frac{1}{2}\int_{t_0}^{t_1}\int_{\mathbb{R}^d} |\partial_t \psi(t,x)|^2 dxdt \quad \textit{and} \\
\mathcal{A}_2(\psi) &\stackrel{\text{def}}{=} -\frac{1}{2}\int_{t_0}^{t_1}\int_{\mathbb{R}^d} F((J\psi(t,x))^{-1}\rho_0(x))J\psi(t,x)dxdt.
\end{aligned}
$$

where $J\psi$ denotes the jacobian determinant of ψ, i.e. $J\psi \stackrel{\text{def}}{=} \det D\psi$.

Remark The term $\mathcal{A}_1$ modelize cinetic energy of the system, the term $\mathcal{A}_2$ of internal energy.

Proposition 3.3.1 *The functionnals $\mathcal{A}_1$ and $\mathcal{A}_2$ are differentiables of $\mathcal{L}$ in $\mathbb{R}$ and we have*

$$
\begin{aligned}
D\mathcal{A}_1(\psi)h &= \int_{t_0}^{t_1}\int_{\mathbb{R}^d} \rho_0(x)\partial_t\psi(t,x)\cdot\partial_t h(t,x)dxdt \quad \textit{and} \\
D\mathcal{A}_2(\psi)h &= \int_{t_0}^{t_1} G((J\psi(t,x))^{-1}\rho_0(x))(\operatorname{div}\tau)(t,\psi(t,x))J\psi(t,x)dtdx \quad \textit{with} \\
G(y) &\stackrel{\text{def}}{=} F(y) - yF'(y) \quad \textit{and} \quad \tau(t,x) \stackrel{\text{def}}{=} h(t,\psi^{-1}(t,x)) \quad \textit{and} \\
\operatorname{div} h(t,x) &= \sum_{j=1}^{d} \frac{\partial}{\partial x_j} h^j(t,x)\,;
\end{aligned}
$$

la differential being defined sur

$$\overrightarrow{\mathcal{L}} \stackrel{\text{def}}{=} \{h \in C^\infty([0,1]\times\mathbb{R}^d;\mathbb{R}^d) \,/\, \forall x \in \mathbb{R}^d\,, \quad h(0,x) = h(1,x) = 0\}.$$

As the functionnal $\mathcal{A}_1$ is quadratic, the computation of $D\mathcal{A}_1$ is trivial. The computation of $D\mathcal{A}_2$ comes from the chain rule. The proof of the following lemma is left to the reader as an exercise.

Lemma 3.3.1 *We have the following formula*

$$DJ(\psi)\cdot h = \sum_{j=1}^{d} \det\Big(D_x\psi^1, \cdots, D_x\psi^{j-1}, D_x h^j, D_x\psi^{j+1}\cdots, D_x\psi^d\Big)\,.$$

If we assume that $\tau(t,x) \stackrel{\text{def}}{=} h(t,\psi^{-1}(t,x))$*, then we have*

$$(DJ(\psi)\cdot h)(t,x) = (\operatorname{div}\tau)(t,\psi(t,x))J(\psi)(t,x).$$

The chain rule implies that

$$D(J\psi)^{-1}h(t,x) = -(J\psi(t,x))^{-1}(\operatorname{div}\tau)(t,\psi(t,x))\,;$$

the proposition is proved.

Now we can define perfect compressible fluids .

Definition 3.3.3 *A perfect compressible fluid is a a fluid whose evolution follows extremals of the functional* $\mathcal{A}$*, i.e. following an element* ψ *of* $\mathcal{L}$ *such that*

$$\forall h \in \overrightarrow{\mathcal{L}}\,,\ D\mathcal{A}(\psi)h = 0.$$

3.4 The eulerian point of view in the compressible case

The above definition seems rather implicit. The purpose of this section is to from a lagrangian description , i.e. a description which is based on the evolution of each pointwise particule x to a eulerian description, i.e. to a description of the fluid based on the knowledge of the speed of the fluid fluid in whole of its points.

Mathematically, the link between these two points of view is the theory of ordinary differential equations. Indeed, let us consider an element ψ of $\mathcal{L}$. Then the associated vector field is defined by

$$v(t,x) = \partial_t\psi(t,\psi^{-1}(t,x)). \tag{3.4}$$

It is clear that v belongs to $\overrightarrow{\mathcal{L}}$ and that ψ is solution of

$$\left\{\begin{array}{rcl} \partial_t\psi(t,x) & = & v(t,\psi(t,x)) \\ \psi(0,x) & = & x. \end{array}\right. \tag{3.5}$$

Conversely, if v belongs to $\overrightarrow{\mathcal{L}}$, Cauchy-Lipschitz Theorem allows to define a flow ψ belonging to $\mathcal{L}$ by the above system (3.5).

Now let us state the theorem which justifies this approach and derives the Euler system of an incompressible fluid.

Theorem 3.4.1 *Let ψ be an extremal of the functionnal $\mathcal{A}$, i.e. an element of $\mathcal{L}$ such that*

$$\forall h \in \overrightarrow{\mathcal{L}}\,,\ D\mathcal{A}(\psi)h = 0.$$

Let us consider time dependant vector field $v$defined by the above relation (3.4); let us define

$$\rho(t,x) \stackrel{\text{def}}{=} \rho_0(\psi^{-1}(t,x))J\psi(t,\psi^{-1}(t,x))^{-1},$$

then the couple (ρ, v) satisfies the following system, called compressible Euler system

$$(E_{comp}) \left\{ \begin{array}{rcl} \partial_t \rho + v\cdot\nabla\rho + \rho \operatorname{div} v & = & 0 \\ \rho(\partial_t v + v\cdot\nabla v) + \nabla p & = & 0 \end{array} \right. \quad \textit{with} \quad v\cdot\nabla a \stackrel{\text{def}}{=} \sum_{j=1}^{d} v^j \frac{\partial a}{\partial x_j} \quad \textit{and} \quad p = G(\rho).$$

The proof is very simple. First, the equation on ρ, which comes form the mass conservation, is nothing more than the translation in terms of partial differential equations of the fact that, by definition of ρ, we have

$$\rho(t,\psi(t,x)) = \rho_0(x)(J\psi(t,x))^{-1}.$$

By time derivation of this formula, it comes frome the chain rule and from Lemma 3.3.1

$$\begin{aligned} (\partial_t\rho + v\cdot\nabla\rho)(t,\psi(t,x)) &= -\rho_0(x)(J\psi(t,x))^{-2}DJ(\psi)(t,\psi(t,x))\cdot\partial_t\psi(t,x) \\ &= -\rho_0(x)(J\psi(t,x))^{-2}DJ(\psi)\cdot v(t,\psi(t,x)) \\ &= -\rho_0(x)(J\psi(t,x))^{-1}(\operatorname{div} v)(t,\psi(t,x)) \\ &= -(\rho \operatorname{div} v)(t,\psi(t,x)). \end{aligned}$$

For the second equation, we use the hypothesis $D\mathcal{A}(\psi)h = 0$. Using an integration by parts and the definition of v and ρ, we find that

$$\begin{aligned} D\mathcal{A}_1(\psi)h &= -\int_{t_0}^{t_1}\int_{\mathbb{R}^d} \rho_0(x)\partial_t^2\psi(t,x)h(t,x)dxdt \\ &= -\int_{t_0}^{t_1}\int_{\mathbb{R}^d} \rho_0(x)\partial_t v(t,\psi(t,x))h(t,x)dxdt \\ &= -\int_{t_0}^{t_1}\int_{\mathbb{R}^d} \rho_0(x)(\partial_t v + v\cdot\nabla v)(t,\psi(t,x))\tau(t,\psi(t,x))dxdt \end{aligned}$$

Performing the change of variable $y = \psi(t,x)$, we find, by definition of ρ,

$$D\mathcal{A}_1(\psi)h = \int_{t_0}^{t_1}\int_{\mathbb{R}^d} \rho(x)(\partial_t v + v\cdot\nabla v)(t,x)\tau(t,x)dxdt. \tag{3.6}$$

Performing the change of variable $y = \psi(t,x)$ into the formula of $D\mathcal{A}_2$, it comes

$$D\mathcal{A}_2(\psi)h = -\int_{t_0}^{t_1}\int_{\mathbb{R}^d} \nabla p(t,x)\tau(t,x)dtdx. \tag{3.7}$$

Applying these two formulas (3.6) and (3.7), we find that

$$D\mathcal{A}(\psi)h = -\int_{t_0}^{t_1}\int_{\mathbb{R}^d} \Big(\rho(x)(\partial_t v + v\cdot\nabla v)(t,x) + \nabla p(t,x)\Big)\tau(t,x)dtdx$$

Thus the theorem is proved.

3.5 The incompressible case

As above, the idea is to define a perfect incompressible fluid as an incompressible fluid whose evolution follows extremals of the functionnal action $\mathcal{A}_1$ restricted to the space $\mathcal{L}_0$. In order to define the notion of infinitesimal variation on the space $\mathcal{L}_0$ which is included in an affine space. Following the classical definition of the tangent space of a submanifold of $\mathbb{R}^d$, we give the following definition.

Definition 3.5.1 *An infinitesimal variation at a point ψ of the space $\mathcal{L}_0$ is the derivative at 0 of any function Θ continuously differentiable from $[0,1]$ into $\mathcal{L}_0$ such that $\Theta(0) = \psi$.*

Because of some regularity problem, it is not very easy to describe exactly the set of infinitesimal variations. Moreover, the intuition of finite dimensionnal spaces must be used with a lot of care always because of regularity problems. The following proposition will be enough for our purpose here.

Proposition 3.5.1 *Let us denote by $\overrightarrow{\mathcal{L}}_0$ the set of vector fields τ whose coefficients are continuously differentiables on $[t_0, t_1] \times \mathbb{R}^d$ and such that*

$$\tau(t_0) = \tau(t_1) = 0 \quad \text{and} \quad \forall\, t \in [t_0, t_1]\,,\ \operatorname{div} \tau(t) = 0.$$

Let θ be an infinitesimal variation at a point ψ of $\mathcal{L}_0$; it exists a vector field τ of the space $\overrightarrow{\mathcal{L}}_0$ such that $\theta(t,x) = \tau(t, \psi(t,x))$.

Conversely, let α a smooth compactly supported function on $]t_0, t_1[$ and τ a divergence free vector field whose componants belong to the space $\mathcal{S}$; if $\theta(t,x) = \alpha(t)\tau(\psi(t,x))$ then θ is an infinitesimal variation at point ψ.

Let θ an infinitesimal variation at point ψ. By definition, it exists a function Θ continuously differentiable of $[0,1]$ in $\mathcal{L}_0$ such that

$$\partial_s \Theta(s,t,x)_{|s=0} = \theta(t,x) \quad \text{et} \quad \Theta(0,t,x) = \psi(t,x). \tag{3.8}$$

As for any s and any t, $\Theta(s,t)$ is a diffeomorphism, we can define a vector field $\widetilde{\tau}(s,t,x)$ by

$$\widetilde{\tau}(s,t,x) = \partial_s \Theta(s,t,\Theta^{-1}(s,t,x)).$$

This means that

$$\partial_s \Theta(s,t,x) = \widetilde{\tau}(s,t,\Theta(s,t,x))$$

Thanks to Lemma 3.3.1, it is enough to apply the chain rule, which gives

$$\partial_s \det D_x \Theta(s,t,x) = \operatorname{div} \widetilde{\tau}(s,t,\Theta(s,t,x)) \times \det(D_x \theta(s,x)). \tag{3.9}$$

As $\det D_x \Theta(s,t,x) = 1$, we have

$$\forall (s,t) \in [0,1] \times [t_0, t_1]\ ,\ \operatorname{div} \widetilde{\tau}(s,t) = 0.$$

As $\partial_s \theta(0,t,x) = \widetilde{\tau}(0,t,\psi(t,x))$, we have the first point of the proposition defining

$$\tau(t,x) \overset{\text{def}}{=} \widetilde{\tau}(0,t,x).$$

The second point is very easy. It is enough to solve the following differential equation

$$\begin{cases} \partial_s \Theta(s,t,x) &= \alpha(t)\tau(t,\Theta(s,t,x)), \\ \Theta(0,t,x) &= \psi(t,x). \end{cases}$$

Now let us define mathematically what a perfect incompressible fluid is.

Definition 3.5.2 *A fluid is perfect and incompressible if its evolution between the time t_0 and situation ψ_1 at time t_1 if its follows an element ψ of $\mathcal{L}_0$ such that, for any infinitesimal variation θ at point ψ, we have $D\mathcal{A}_1(\psi)\cdot\theta = 0$.*

The above definition can be formulated as the fact perfect incompressible fluids follows extremales of the action functionnal action, which is defined on the space of curves of measure preserving diffeomorphisms.

The above description of the evolution of an incompressible fluid by a curve on the spaceof measure preserving diffeomorphisms; it is the lagrangien point of view. Let us take now the Eulerian point of view.

Theorem 3.5.1 *Let ψ an evolution of a pecfect imcompressible fluid and v the divergence free vector field associated with ψ by (3.4). It exists then a distribution tempérée p such that, if l'on pose $v\cdot\nabla = \sum_{i=1}^{d} v^i\partial_i$, we have*

$$\partial_t v + v\cdot\nabla v = -\nabla p.$$

Let us suppose that ψ is an evolution of a perfect incompressible fluid. Definition 3.5.1 and Proposition 3.5.1 imply that, for any $\alpha \in \mathcal{D}(]t_0,t_1[)$ and any divergence free vector field, we have $\tau \in C^\infty([t_0,t_1];\mathcal{S}(\mathbb{R}^d;\mathbb{R}^d))$,

$$\int_{t_0}^{t_1}\int_{\mathbb{R}^d} \partial_t\psi(t,x)\partial_t(\alpha(t)\tau(t,\psi(t,x)))dtdx = 0.$$

Form (3.5), it comes

$$\int_{t_0}^{t_1}\int_{\mathbb{R}^d} v(t,\psi(t,x))\partial_t(\alpha(t)\tau(t,\psi(t,x)))dtdx = 0.$$

An integration by parts with respect the time variable ensures that

$$\int_{t_0}^{t_1}\int_{\mathbb{R}^d} \partial_t(v(t,\psi(t,x)))\alpha(t)\tau(t,\psi(t,x))dtdx = 0.$$

As $\partial_t(v(t,\psi(t,x)) = (\partial_t v + v\cdot\nabla v)(t,\psi(t,x))$, and as $\psi(t)$ is a measure preserving diffeomorphism, we have, for any $\alpha \in \mathcal{D}(]t_0,t_1[)$ and any divergence free vector field τ,

$$\int_{t_0}^{t_1}\int_{\mathbb{R}^d} (\partial_t v + v\cdot\nabla v)(t,x)\alpha(t)\tau(t,x)dtdx = 0. \tag{3.10}$$

As $\partial_t v + v\cdot\nabla v$ is a continuous function of the time, we have, for any divergence free vector field τ and for any time t,

$$\int_{\mathbb{R}^d} (\partial_t v + v\cdot\nabla v)(t,x)\tau(t,x)dx = 0. \tag{3.11}$$

The first part of the theorem will be a consequence of the following proposition, which is classical in the theory ofdistributions.

Proposition 3.5.2 *Let w a vector field the coefficients of which are tempered distribtions. The existence of a tempered distribution p such that $w = \nabla p$ is equivalent to the fact that the curl of w is 0, i.e. $\partial_j w^i = \partial_i w^j$.*

The proof of this proposition is a exercice of theory of distributions. Let us consider f_0 a smooth compactly supported function ofone real variable of integral 1. Let us assume that the dimension d is equal to 1. Then let us define, for $\phi \in \mathcal{D}(\mathbb{R})$,

$$\Phi(\phi)(x) = \int_{-\infty}^{x} \Big(\phi(y) - \Big(\int_{-\infty}^{\infty} \phi(y')dy'\Big) f_0(y)\Big)\, dy.$$

It is clear that $\Phi(\partial_x \phi) = \phi$ and that $\Phi(\phi)$ is a smooth compactly supported function. Morever, the map Φ defined above is a linear continuous map from $\mathcal{S}(\mathbb{R})$ into itsself. Indeed, if $x \leq -A$, with A strictement positif, then

$$|\Phi(\phi)(x)| \leq \int_{-\infty}^{x} |\phi(y)|dy + \Big|\int_{-\infty}^{\infty} \phi(y')dy'\Big| \int_{-\infty}^{x} |f_0(y)|dy.$$

If $-x$ is strictement positif, we have, for any integer N greater that 2,

$$|\Phi(\phi)(x)| \leq \int_{-x}^{\infty} \Big(\frac{\sup_{y\in\mathbb{R}} (1+|y|)^{N+1} |\phi(y)|}{(1+|y'|)^{N+1}} + \|\phi\|_{L^1} \frac{\sup_{y\in\mathbb{R}} (1+|y|)^{N+1} |f_0(y)|}{(1+|y'|)^{N+1}} \Big) dy'.$$

Let us define $\mathcal{N}_N(\phi) = \|\phi\|_{L^1} + \sup_{y\in\mathbb{R}^d} |(1+|y|)^{N+1}\phi(y)|$. We have

$$|x|^N |\Phi(\phi)(x)| \leq C\mathcal{N}_N(\phi).$$

As

$$\int_{-\infty}^{\infty} \Big(\phi(y) - \Big(\int_{-\infty}^{\infty} \phi(y')dy'\Big) f_0(y)\Big)\, dy = 0,$$

the same thing holds for positive x. The fact that

$$\partial_x \Phi(\phi) = \phi - f_0 \int_{-\infty}^{\infty} \phi(y')dy'$$

concludes the proof of the continuity of Φ on $\mathcal{S}(\mathbb{R})$.

Let w a tempered distribution on $\mathbb{R}$. Let us define p by

$$< p, \phi >= - < w, \Phi(\phi) > .$$

As Φ is a linear continuous map from $\mathcal{S}(\mathbb{R})$ into itsself, p is a tempered distribution. Moreover,

$$< \partial_x p, \phi >= - < p, \partial_x \phi >=< w, \Phi(\partial_x \phi) >=< w, \phi > .$$

Thus the result holds in dimension $d = 1$.

Let us do an induction by assuming that the result holds in dimension $d \leq k-1$. Let w a vector field the coefficients of which are tempered distribtions on $\mathbb{R}^k$ and satisfies the hypothesis on the proposition.Thanks to the induction hypothesis, a tenpered distribution π on $\mathbb{R}^{k-1}$ exists such that

$$< \partial_i \pi, \theta >=< w^i, f_0 \otimes \theta > .$$

Then let us define the tempered distribution on $\mathbb{R}^k$ by

$$< p, \phi >= - < w^1, \Phi_k(\phi) > + < \pi, \int_{-\infty}^{\infty} \phi(y_1)dy_1 >,$$

with $\Phi_k(\phi)(x) = \Phi(\phi(\cdot, x_2, \dots, x_k))(x_1)$. Now it is enough to check that $\partial_i p = w^i$. First, let us observe that

$$< \partial_1 p, \phi >=< w^1, \Phi_k(\partial_1 \phi) > - < \pi, \int_{-\infty}^{\infty} \partial_1 \phi(y_1) dy_1 >=< w^1, \phi > .$$

If $i \geq 2$, we have

$$\begin{aligned} < \partial_i p, \phi > &= + < w^1, \Phi_k(\partial_i \phi) > - < \pi, \int_{-\infty}^{\infty} \partial_i \phi(y_1) dy_1 > \\ &= - < \partial_i w^1, \Phi_k(\phi) > + < \partial_i \pi, \int_{-\infty}^{\infty} \phi(y_1) dy_1 > \\ &= - < \partial_1 w^i, \Phi_k(\phi) > + < w^i, f_0 \otimes \int_{-\infty}^{\infty} \phi(y_1) dy_1 > \\ &= < w^i, \partial_1 \Phi_k(\phi) > + < w^i, f_0 \otimes \int_{-\infty}^{\infty} \phi(y_1) dy_1 > \\ &= < w^i, \phi > \end{aligned}$$

Thus the proposition.

We can easily deduce the two following corollaries :

Corollary 3.5.1 *Let w a vector fields the coefficients of which are tempered distribtions such that, for any divergence free vector field u of $\mathcal{S}(\mathbb{R}^d;\mathbb{R}^d)$, we have*

$$< w, u >= \sum_i < w^i, u^i >= 0.$$

A tempered distribution p exists such that $w = \nabla p$.

Corollary 3.5.2 *Let w a divergence free vector field on $\mathbb{R}^2$ whose coefficients are tempered distributions. It exists a tempered distribution f such that*

$$w \stackrel{\text{def}}{=} \nabla^\perp f = (-\partial_2 f, \partial_1 f).$$

In order to prove the first corollary, let us consider a function ϕ of the space $\mathcal{S}(\mathbb{R}^d)$. In the case when i and j are two distinct positive integers less or equal to d, let us consider the vector field u whose component of index j is $\partial_i \phi$, whose component of index i is $-\partial_j \phi$ the others being identically 0. It is clear that u is a divergence free vector field. Thus by hypothesis

$$< w, u >=< w^j, \partial_i \phi > - < w^i, \partial_j \phi >=< \partial_j w^i - \partial_i w^j, \phi >= 0.$$

It turns out that $\partial_j w^i - \partial_i w^j = 0$ and Corollary 3.5.1 is proved.

In order to prove the second corollary, let us consider the divergence free vector field $\widetilde{w} = (-w^2, w^1)$. it is clear that we have $\partial_1 \widetilde{w}^2 - \partial_2 \widetilde{w}^1 = \operatorname{div} w = 0$. It exists donc a tempered distribution f such that $\widetilde{w} = (\partial_1 f, \partial_2 f)$. Thus le corollary 3.5.2.

Let us go back to the proof of Theorem 3.5.1. It is clear from Corollary 3.5.1 and from (3.10) that, if ψ is a evolution of a perfect incompressible fluid, then the vector field associated with $\psi(t)$ throug (3.4) satisfies

$$\partial_t v + v \cdot \nabla v = -\nabla p. \tag{3.12}$$

Conversely, if v is a vector field satisfying (3.12). Let us consider the flow ψ of v defined by (3.5) et θ an infinitesimal variation in ψ. Then, we have

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^d} \partial_t \psi(t,x) \partial_t \theta(t,x) dt dx = -\int_{t_0}^{t_1} \int_{\mathbb{R}^d} \partial_t^2 \psi(t,x) \theta(t,x) dt dx.$$

Proposition 3.5.1 ensures the existence of a divergence free vector field τ such that $\theta = \tau(t, \psi(t,x))$. It turns out that

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^d} \partial_t \psi(t,x) \partial_t \theta(t,x) dt dx = \int_{t_0}^{t_1} \int_{\mathbb{R}^d} (\nabla p)(\psi(t,x)) \tau(t, \psi(t,x)) dt dx.$$

For any time t, the diffeomorphism $\psi(t)$ preserves the measure; this concludes the proof the theorem.

Now let us give a weak formulation to Equation (3.12).This formulation is equivalent to the one of Relation (3.12) in the case when the vector field solution is smooth enough. Nevertheless, it can be important to have a weak formulation. If v is a divergence free vector field continuously differentiable, we have $v \cdot \nabla a = \operatorname{div}(av)$ for any function a continuously differentiable. This gives

$$\partial_t v + v \cdot \nabla v = \partial_t v + \operatorname{div} v \otimes v,$$

where $\operatorname{div} v \otimes v$ denotes the vector field whose coordiante of order j is $\displaystyle\sum_{j=1}^{d} \partial_j (v^i v^j)$. Thus we get the following formulation of incompressible Euler equations.

$$(E) \begin{cases} \partial_t v + \operatorname{div} v \otimes v &= -\nabla p \\ \operatorname{div} v &= 0 \\ v_{|t=0} &= v_0. \end{cases}$$

Remarks

- The proofs of sections 3.1 and 3.2 must be known. The sections 3.3, 3.4 and 3.5 are important in the point of view of modelization. It is not necessary to know the proofs of these sections.

Chapter 4

Leray's Theorem on Navier-Stokes equations

In this chapitre, we shall prove the existence of global solutions for the incompressible Navier-Stokes system in a bounded domain with Dirichlet boundary conditions which means

$$\begin{cases} \partial_t v + v \cdot \nabla v - \nu \Delta v &= -\nabla p \\ \operatorname{div} v &= 0 \\ v_{|t=0} &= v_0 \\ v_{|\partial\Omega} &= 0. \end{cases}$$

Before studying this problem, we study a simplier one, called "time dependent Stokes problem".

4.1 The time dependent Stokes problem

Given a positive viscosity ν, the evolution Stokes problem reads as follows:

$$(ES_\nu) \begin{cases} \partial_t u - \nu \Delta u &= f - \nabla p \\ \operatorname{div} u &= 0 \\ u_{|\partial\Omega} &= 0 \\ u_{|t=0} &= u_0 \in \mathcal{H}. \end{cases}$$

Let us define what a solution of this problem is.

Definition 4.1.1 *Let u_0 be in $\mathcal{H}$ and f in $L^2_{loc}(\mathbb{R}^+; \mathcal{V}')$. We shall say that u is a solution of (ES_ν) with initial data u_0 and external force f if and only if u belongs to the space*

$$C(\mathbb{R}^+; \mathcal{V}'_\sigma) \cap L^\infty_{loc}(\mathbb{R}^+; \mathcal{H}) \cap L^2_{loc}(\mathbb{R}^+; \mathcal{V}_\sigma)$$

and satisfies, for any Ψ in $C^1(\mathbb{R}^+; \mathcal{V}_\sigma)$,

$$\langle u(t), \Psi(t) \rangle + \int_{[0,t]\times\Omega} \Big(\nu \nabla u : \nabla \Psi - u \cdot \partial_t \Psi \Big)(t', x)\, dx dt' = \int_\Omega u_0(x) \cdot \Psi(0, x)\, dx + \int_0^t \langle f(t'), \Psi(t') \rangle\, dt'.$$

The following theorem holds.

Theorem 4.1.1 *The problem (ES_ν) has a unique solution in the sense of the above definition. Moreover this solution belongs to $C(\mathbb{R}^+;\mathcal{H})$ and satisfies*

$$\frac{1}{2}\|u(t)\|_{L^2}^2+\nu\int_0^t\|\nabla u(t')\|_{L^2}^2dt'=\frac{1}{2}\|u_0\|_{L^2}^2+\int_0^t\langle f(t'),u(t')\rangle dt'.$$

Proof of Theorem 4.1.1. In order to prove uniqueness, let us consider some function u in $C(\mathbb{R}^+;\mathcal{V}'_\sigma)\cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$ such that, for all Ψ in $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$,

$$\langle u(t),\Psi(t)\rangle+\int_0^t\int_\Omega\Big(\nu\nabla u:\nabla\Psi-u\cdot\partial_t\Psi\Big)(t',x)\,dxdt'=0.$$

This is valid in particular for the time independent function $\Psi(t)\equiv e_k$ where the family vector fields $(e_k)_{k\in\mathbb{N}}$ is given by Theorem 3.2.2. This gives

$$\begin{aligned}\langle u(t),e_k\rangle &= -\nu\int_0^t\int_\Omega\nabla u(t',x):\nabla e_k(x)dxdt'\\ &= \nu\int_0^t\langle u(t'),\Delta\,\mathbb{P}_k\,\Psi\rangle.\end{aligned}$$

Thanks to the spectral Theorem 3.2.2 together with the fact that, for almost every t', $u(t)$ belongs to $\mathcal{V}_\sigma$, we have

$$-\int_\Omega\nabla u(t',x):\nabla e_k(x)dx=\langle\Delta e_k,u(t')\rangle=\lambda_k^2\langle e_k,u(t')\rangle.$$

This gives

$$\langle u(t),e_k\rangle=\int_0^t\lambda_k^2\langle e_k,u(t')\rangle dt'.$$

The fact that $\langle u(0),e_k\rangle=0$ implies that, for any k, $\langle u(t),e_k\rangle=(u|e_k)_\mathcal{H}=0$. Thus $u\equiv0$.

In order to prove existence, let us consider a sequence $(f_k)_{k\in\mathbb{N}}$ associated with f by Lemma 4.3.1 page 43 and then the approximated problem

$$(ES_{\nu,k})\left\{\begin{array}{rcl}\partial_t u_k-\nu\,\mathbb{P}_k\,\Delta u_k &=& f_k\\ u_{k|t=0} &=& \mathbb{P}_k\,u_0\end{array}\right.\quad\text{with}\quad \mathbb{P}_k\,f\overset{\text{def}}{=}\sum_{j\leq k}\langle f,e_j\rangle e_j. \tag{4.1}$$

Again thanks to Theorem 3.2.2 page 30, it is a linear ordinary differential equation on $\mathcal{H}_k$ which has a global solution u_k which is $C^1(\mathbb{R}^+;\mathcal{H}_k)$. By an energy estimate in $(ES_{\nu,k})$ we get that

$$\frac{1}{2}\frac{d}{dt}\|u_k(t)\|_{L^2}^2+\nu\|\nabla u_k(t)\|_{L^2}^2=\langle f_k(t),u_k(t)\rangle.$$

A time integration gives

$$\frac{1}{2}\|u_k(t)\|_{L^2}^2+\nu\int_0^t\|\nabla u_k(t')\|_{L^2}^2dt'=\frac{1}{2}\|\,\mathbb{P}_k\,u(0)\|_{L^2}^2+\int_0^t\langle f_k(t'),u_k(t')\rangle dt'. \tag{4.2}$$

In order to pass to the limit, we write an energy estimate for $u_k-u_{k+\ell}$, which gives

$$\begin{aligned}\delta_{k,\ell}(t) &\overset{\text{def}}{=} \frac{1}{2}\|(u_k-u_{k+\ell})(t)\|_{L^2}^2+\nu\int_0^t\|\nabla(u_k-u_{k+\ell})(t')\|_{L^2}^2dt'\\ &= \frac{1}{2}\|(\mathbb{P}_k-\mathbb{P}_{k+\ell})u(0)\|_{L^2}^2+\int_0^t\langle(f_k-f_{k+\ell})(t'),u_k(t')\rangle dt'\\ &\leq \frac{1}{2}\|(\mathbb{P}_k-\mathbb{P}_{k+\ell})u(0)\|_{L^2}^2+\frac{C}{\nu}\int_0^t\|(f_k-f_{k+\ell})(t')\|_{\mathcal{V}'_\sigma}^2dt'\\ &\quad+\frac{\nu}{2}\int_0^t\|\nabla(u_k-u_{k+\ell})(t')\|_{L^2}^2dt'+\frac{\nu}{2}\int_0^t\|(u_k-u_{k+\ell})(t')\|_{L^2}^2dt'.\end{aligned}$$

Using Poincaré's inequality, this implies that

$$\delta_{k,\ell}(t) \leq \|(\mathbb{P}_k - \mathbb{P}_{k+\ell})u(0)\|_{L^2}^2 + \frac{C}{\nu}\int_0^t \|(f_k - f_{k+\ell})(t')\|_{\mathcal{V}'_\sigma}^2 dt'.$$

This implies immediately that the sequence $(u_k)_{k\in\mathbb{N}}$ is a Cauchy one in the space $C(\mathbb{R}^+;\mathcal{H})\cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$. Let us denote by u the limit and prove that u is a solution in the sense of Definition 4.1.1. As u_k is a C^1 solution of the ordinary differential equation $(ES_{\nu,k})$, we have, for a Ψ in $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$,

$$\frac{d}{dt}\langle u_k(t),\Psi(t)\rangle = \nu\langle\Delta u_k(t),\Psi(t)\rangle + \langle f_k(t),\Psi(t)\rangle + \langle u_k(t),\partial_t\Psi(t)\rangle.$$

By time integration, we get

$$\begin{aligned}\langle u_k(t),\Psi(t)\rangle = -\nu\int_0^t\int_\Omega \nabla u_k(t',x):\nabla\Psi(t',x)\,dxdt'\\ + \int_0^t\langle f_k(t'),\Psi(t')\rangle dt' + \langle\mathbb{P}_k\,u(0),\Psi(0)\rangle + \int_0^t\langle u_k(t'),\partial_t\Psi(t')\rangle dt'.\end{aligned}$$

Passing to the limit in the above equality and in (4.2) gives the theorem.

Remark. The solution is given by the explicit formula

$$\begin{aligned} u(t) &= \sum_{j\in\mathbb{N}} U_j(t)e_j \quad \text{with}\\ U_j(t) &\stackrel{\text{def}}{=} e^{-\nu\mu_j^2 t}(u_0|e_j)_{L^2} + \int_0^t e^{-\nu\mu_j^2(t-t')}\langle f(t'),e_j\rangle\,dt'. \end{aligned} \tag{4.3}$$

In the case of the whole space $\mathbb{R}^d$, we have the following analogous formula

$$\begin{aligned} u(t,x) &= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\widehat{u}(t,\xi)d\xi \quad \text{with}\\ \widehat{u}(t,\xi) &\stackrel{\text{def}}{=} e^{-\nu|\xi|^2 t}\widehat{u}_0(\xi) + \int_0^t e^{-\nu|\xi|^2(t-t')}\widehat{f}(t',\xi)\,dt'. \end{aligned} \tag{4.4}$$

4.2 The concept of weak solution

Let us state now the weak formulation of the incompressible Navier–Stokes system (NS_ν).

Definition 4.2.1 *Given a domain Ω in $\mathbb{R}^d$, we shall say that u is a weak solution of the Navier–Stokes equations on $\mathbb{R}^+\times\Omega$ with an initial data u_0 in $\mathcal{H}$ and an external force f in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$ if and only if u belongs to the space*

$$C(\mathbb{R}^+;\mathcal{V}'_\sigma)\cap L^\infty_{loc}(\mathbb{R}^+;\mathcal{H})\cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$$

and for any function Ψ in $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$, the vector field u satisfies the following condition:

$$\begin{aligned}\int_\Omega (u\cdot\Psi)(t,x)\,dx + \int_0^t\int_\Omega\Big(\nu\nabla u:\nabla\Psi - u\otimes u:\nabla\Psi - u\cdot\partial_t\Psi\Big)(t',x)\,dxdt'\\ = \int_\Omega u_0(x)\cdot\Psi(0,x)\,dx + \int_0^t\langle f(t'),\Psi(t')\rangle\,dt' \quad \textit{with}\\ \nabla u:\nabla\Psi = \sum_{j,k=1}^d \partial_j u^k\partial_j\Psi^k \quad \textit{and} \quad u\otimes u:\nabla\Psi = \sum_{j,k=1}^d u^j u^k\partial_j\Psi^k.\end{aligned}$$

Let us remark that the above relation means that the equality in (NS_ν) must be understood as an equality in the sense of $\mathcal{V}'_\sigma$. Now let us state the Leray theorem.

Theorem 4.2.1 *Let Ω be a domain of $\mathbb{R}^d$ and u_0 a vector field in $\mathcal{H}$. Then, there exists a global weak solution u to (NS_ν) in the sense of Definition 4.2.1. Moreover, this solution satisfies the energy inequality for all $t \geq 0$,*

$$\begin{aligned}\frac{1}{2}\int_\Omega |u(t,x)|^2 dx \quad &+ \quad \nu\int_0^t\int_\Omega |\nabla u(t',x)|^2 dx dt' \\ &\leq \quad \frac{1}{2}\int_\Omega |u_0(x)|^2 dx + \int_0^t \langle f(t',\cdot), u(t',\cdot)\rangle dt'. \qquad (4.5)\end{aligned}$$

It is convenient to state the following definition.

Definition 4.2.2 *A solution of (NS_ν) in the sense of the above Definition 4.2.1 which moreover satisfies the energy inequality (4.5) is called a Leray solution of (NS_ν).*

Let us remark that the energy inequality implies a control on the energy.

Proposition 4.2.1 *Any Leray solution u of (NS_ν) satisfies*

$$\|u(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla u(t')\|_{L^2}^2 dt' \leq \|u_0\|_{L^2}^2 + \frac{C}{\nu}\int_0^t \|f(t')\|_{\mathcal{V}'_\sigma}^2 dt'.$$

Proof of Proposition 4.2.1 By definition of the norm $\|\cdot\|_{\mathcal{V}'_\sigma}$, we have

$$\langle f(t,\cdot), u(t,\cdot)\rangle \leq \|f(t,\cdot)\|_{\mathcal{V}'_\sigma}\|u(t,\cdot)\|_{\mathcal{V}_\sigma}.$$

Inequality (4.5) becomes

$$\|u(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla u(t')\|_{L^2}^2 dt' \leq \|u_0\|_{L^2}^2 + \int_0^t \|f(t')\|_{\mathcal{V}'_\sigma}\|u(t',\cdot)\|_{\mathcal{V}}^2 dt'.$$

As $\|u(t',\cdot)\|_{\mathcal{V}}^2 = \|\nabla u(t',\cdot)\|_{L^2}^2$, we get, using the fact that $2ab \leq a^2+b^2$,

$$\|u(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla u(t')\|_{L^2}^2 dt' \leq \|u_0\|_{L^2}^2 + \frac{C}{\nu}\int_0^t \|f(t')\|_{\mathcal{V}'_\sigma}^2 dt'.$$

Thus the proposition is proved.

The outline of this section is now the following:

- first approximate solutions are built in spaces with finite frequencies by using simple ordinary differential equations results in L^2–type spaces.
- Next, a compactness result is derived.
- Finally the conclusion is obtained by passing to the limit in the weak formulation, taking especially care of the nonlinear terms.

4.3 Construction of approximate solutions

In this section, we intend to build approximate solutions of the Navier–Stokes equations. We use the projections $\mathbb{P}_k$ defined in (4.1) and denote by $\mathcal{H}_k$ the space $\mathbb{P}_k \mathcal{H} = \mathbb{P}_k \mathcal{V}'$.

Lemma 4.3.1 *For any bulk force f in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$, a sequence $(f_k)_{k\in\mathbb{N}}$ exists in $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$ such that for any $k \in \mathbb{N}$ and for any $t > 0$, the vector field $f_k(t)$ belongs to $\mathcal{H}_k$, and*

$$\lim_{k\to\infty} \|f_k - f\|_{L^2([0,T];\mathcal{V}'_\sigma)} = 0.$$

Proof of Lemma 4.3.1: Thanks to Theorem 3.2.2 and to the Lebesgue Theorem, a sequence $(\widetilde{f}_k)_{k\in\mathbb{N}}$ exists in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$ such that for any positive integer k and for almost all positive t, the vector field $\widetilde{f}_k(t)$ belongs to $\mathcal{H}_k$ and

$$\forall T > 0\,,\ \lim_{k\to\infty} \|\widetilde{f}_k - f\|_{L^2([0,T];\mathcal{V}'_\sigma)} = 0.$$

A standard (and omitted) time regularization procedure concludes the proof of the lemma.

In order to construct the approximate solution, let us study the non linear term.

Definition 4.3.1 *Let us define the bilinear map*

$$Q \left\{ \begin{array}{ccc} \mathcal{V}\times\mathcal{V} & \to & \mathcal{V}' \\ (u,v) & \mapsto & -\operatorname{div}(u\otimes v). \end{array} \right.$$

Sobolev embeddings stated in Theorem 2.2.1 ensure that Q is continuous: in the sequel, the following lemma will be useful.

Lemma 4.3.2 *For any u and v in $\mathcal{V}$, the following estimates hold. For d in $\{2,3\}$, a constant C exists such that, for any $\varphi \in \mathcal{V}$,*

$$\langle Q(u,v),\varphi\rangle \le C\|\nabla u\|_{L^2}^{\frac{d}{4}}\|\nabla v\|_{L^2}^{\frac{d}{4}}\|u\|_{L^2}^{1-\frac{d}{4}}\|v\|_{L^2}^{1-\frac{d}{4}}\|\nabla\varphi\|_{L^2}.$$

Moreover for any u in $\mathcal{V}_\sigma$ and any v in $\mathcal{V}$, $\langle Q(u,v),v\rangle = 0$.

Proof of Lemma 4.3.2. The first two inequalities follow directly from Gagliardo–Nirenberg's inequality stated in Corollary 2.2.1, once noticed that

$$\begin{aligned} \langle Q(u,v),\varphi\rangle &\le \|u\otimes v\|_{L^2}\|\nabla\varphi\|_{L^2} \\ &\le \|u\|_{L^4}\|v\|_{L^4}\|\nabla\varphi\|_{L^2}. \end{aligned}$$

In order to prove the third assertion, let us assume that u and v are two vector fields the components of which belong to $\mathcal{D}(\Omega)$. Then we deduce from integrations by parts that

$$\begin{aligned} \langle Q(u,v),v\rangle &= -\int_\Omega (\operatorname{div}(u\otimes v)\cdot v)(x)\,dx \\ &= -\sum_{\ell,m=1}^{d}\int_\Omega \partial_m(u^m(x)v^\ell(x))v^\ell(x)\,dx \\ &= \sum_{\ell,m=1}^{d}\int_\Omega u^m(x)v^\ell(x)\partial_m v^\ell(x)\,dx \\ &= -\int_\Omega |v(x)|^2\operatorname{div} u(x)\,dx - \langle Q(u,v),v\rangle. \end{aligned}$$

Thus, we have

$$\langle Q(u,v),v\rangle = -\frac{1}{2}\int_\Omega |v(x)|^2 \operatorname{div} u(x)\,dx.$$

The two expressions are continuous on $\mathcal{V}$ and by definition, $\mathcal{D}$ is dense in $\mathcal{V}$. Thus the formula is true for any $(u,v)\in\mathcal{V}_\sigma\times\mathcal{V}$, which completes the proof.

Thanks to Theorem 3.2.2 and to the above lemma, we can define $F_k(u) \stackrel{\text{def}}{=} \mathbb{P}_k\, Q(u,u)$. Now let us introduce the following ordinary differential equation

$$(NS_{\nu,k})\quad \begin{cases} \dot u_k(t) &= \nu\,\mathbb{P}_k\,\Delta u_k(t) + F_k(u_k(t)) + f_k(t)\\ u_k(0) &= \mathbb{P}_k\, u_0. \end{cases}$$

Theorem 3.2.2 implies that $\mathbb{P}_k\,\Delta$ is a linear map from $\mathcal{H}_k$ into itself. Thus the continuity properties on Q and $\mathbb{P}_k$ allow to apply the Cauchy–Lipschitz theorem. This gives the existence of $T_k\in]0,+\infty]$ and a unique maximal solution u_k of $(NS_{\nu,k})$ in $C^\infty([0,T_k[;\mathcal{H}_k)$. In order to prove that $T_k=+\infty$, let us observe that, thanks to Lemma 4.3.2 and Theorem 3.2.2

$$\|\dot u_k(t)\|_{L^2} \le \nu\lambda_k\|u_k(t)\|_{L^2} + C\lambda_k^{\frac{d}{4}}\|u_k(t)\|_{L^2}^2 + \|f_k(t)\|_{L^2}.$$

If $\|u_k(t)\|_{L^2}$ remains bounded on some interval $[0,T[$, so does $\|\dot u_k(t)\|_{L^2}$. Thus, for any k, the function u_k satisfies the Cauchy criteria when t tends to T. Thus the solution can be extended beyond T. It follows that a uniform bound on $\|u_k(t)\|_{L^2}$ will imply that $T_k=+\infty$.

4.4 A priori bounds

The purpose of this paragraph is the proof of the following proposition.

Proposition 4.4.1 *The sequence $(u_k)_{k\in\mathbb{N}}$ in bounded in the space*

$$L^\infty_{loc}(\mathbb{R}^+;\mathcal{H})\cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)\cap L^{\frac{8}{d}}_{loc}(\mathbb{R}^+\, L^4(\Omega)).$$

Moreover, the sequence $(\Delta u_k)_{k\in\mathbb{N}}$ is bounded in the space $L^2_{loc}(\mathbb{R}^+;\mathcal{V}'_\sigma)$.

Proof of Proposition 4.4.1 Let us now estimate the L^2 norm of $u_k(t)$. Taking the L^2 scalar product of equation $(NS_{\nu,k})$ with $u_k(t)$, we get

$$\frac{1}{2}\frac{d}{dt}\|u_k(t)\|_{L^2}^2 = \nu(\Delta u_k(t)|u_k(t))_{L^2} + (F_k(u_k(t))|u_k(t))_{L^2} + (f_k(t)|u_k(t))_{L^2}.$$

By definition of F_k, Lemma 4.3.2 implies that

$$(F_k(u_k(t))|u_k(t))_{L^2} = \langle Q(u_k(t),u_k(t)),u_k\rangle = 0.$$

Thus we infer that

$$\frac{1}{2}\frac{d}{dt}\|u_k(t)\|_{L^2}^2 + \nu(\nabla u_k(t)|\nabla u_k(t))_{L^2} = (f_k(t)|u_k(t))_{L^2}. \tag{4.6}$$

By time integration, we get the fundamental energy estimate for the approximate Navier–Stokes system: for all $t\in[0,T_k)$

$$\frac{1}{2}\|u_k(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla u_k(t')\|_{L^2}^2 dt' = \frac{1}{2}\|u_k(0)\|_{L^2}^2 + \int_0^t (f_k(t')|u_k(t'))_{L^2} dt'. \tag{4.7}$$

Using the (well known) fact that $2ab \leq a^2 + b^2$, we get

$$\|u_k(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla u_k(t')\|_{L^2}^2 \, dt' \leq \|u_k(0)\|_{L^2}^2 + \frac{C}{\nu} \int_0^t \|f_k(t')\|_{\mathcal{V}'_\sigma}^2 dt'$$

Gronwall's lemma implies that $(u_k)_{k\in\mathbb{N}}$ remains uniformly bounded in $\mathcal{H}$ for all time, hence that $T_k = +\infty$. In addition, the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in the space $L^\infty_{loc}(\mathbb{R}^+;\mathcal{H}) \cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$. Using Gagliardo-Nirenberg inequalities (see Corollary 2.2.1 page 20), we deduce that the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in the space

$$L^\infty_{loc}(\mathbb{R}^+;\mathcal{H}) \cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma) \cap L^{\frac{8}{d}}_{loc}(L^4(\Omega)).$$

Moreover, we have, for any $v \in \mathcal{V}_\sigma$,

$$\langle -\Delta u_k, v\rangle = (\nabla u_k | \nabla v)_{L^2} \leq \|u_k\|_{H^1_0} \|v\|_{\mathcal{V}}.$$

By definition of the norm $\|\cdot\|_{\mathcal{V}'_\sigma}$, we infer that the sequence $(\Delta u_k)_{k\in\mathbb{N}}$ is bounded in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}'_\sigma)$. The whole proposition is proved.

4.5 Compactness properties

Let us now prove the following fundamental result.

Proposition 4.5.1 *A vector field u exists in $L^\infty_{loc}(\mathbb{R}^+;\mathcal{H}) \cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$ such that up to an extraction (which we omit to note) we have for any positive real number T, for all vector fields $\Psi \in L^2([0,T];\mathcal{V})$,*

$$\lim_{k\to\infty} \int_{[0,T]\times\Omega} (\nabla u_k(t,x) - \nabla u(t,x)) : \nabla\Psi(t,x) \, dtdx = 0. \tag{4.8}$$

$$\lim_{k\to\infty} \int_{[0,T]\times\Omega} |u_k(t,x) - u(t,x)|^2 \, dtdx = 0. \tag{4.9}$$

$$\lim_{k\to\infty} \|u_k - u\|_{L^\infty([0,T];\mathcal{V}'_\sigma)} = 0. \tag{4.10}$$

Proof of Proposition 4.5.1. A standard diagonal process (omitted) with an increasing sequence of positive real numbers T_n reduces the proof of (4.8)–(4.10) to the proof of the same result for any time interval $[0,T]$. The relative weak compactness of the sequence $(u_k)_{k\in\mathbb{N}}$ in the Hilbert space $L^2([0,T];\mathcal{V}_\sigma)$is obvious. Thus (4.8) is true with u in $L^2([0,T];\mathcal{V}_\sigma)$.

In order to prove (4.9), let us establish th at

$$\forall\varepsilon\,,\ \exists k_0 \,/\, \forall k\,,\ \|u_k - \mathbb{P}_{k_0} u_k\|_{L^2([0,T]\times\Omega)} + \|u_k - \mathbb{P}_{k_0} u_k\|_{L^\infty([0,T];\mathcal{V}'_\sigma)} < \frac{\varepsilon}{2}\cdot \tag{4.11}$$

The proof of the claim is based on Theorem 3.2.2. Using this result, we can write that

$$\begin{aligned}\|u_k - \mathbb{P}_{k_0} u_k\|^2_{L^2([0,T];\mathcal{H})} &= \int_0^T \sum_{j\geq k_0+1} \langle u_k(t), e_j\rangle)^2 dt\\ &= \int_0^T \sum_{j\geq k_0+1} \lambda_j^{-2}\lambda_j^2 \langle u_k(t), e_j\rangle)^2 dt.\end{aligned}$$

Using that the sequence $(\lambda_j)_j$ is increasing, we get, by Theorem 3.2.2,

$$\begin{aligned}\|u_k - \mathbb{P}_{k_0} u_k\|^2_{L^2([0,T];\mathcal{H})} &\leq \lambda_{k_0+1}^{-2} \int_0^T \sum_j \lambda_j^2 \langle u_k(t), e_j\rangle)^2 dt \\ &\leq \lambda_{k_0+1}^{-2} \|u_k\|^2_{L^2([0,T];\mathcal{V}_\sigma)}.\end{aligned}$$

Following the same lines we get

$$\|u_k - \mathbb{P}_{k_0} u_k\|^2_{L^2([0,T];\mathcal{V}'_\sigma)} \leq \lambda_{k_0+1}^{-2} \|u_k(t)\|^2_{\mathcal{H}}.$$

The fact that $\lim\limits_{k\to\infty} \lambda_k = +\infty$ ensures (4.11).

Now, let us prove that the sequence $(\mathbb{P}_{k_0} u_k)_k$ is relativement compact in $\mathcal{H}_{k_0}$. Let us notice that $\mathcal{H}_{k_0} = \mathbb{P}_{k_0} \mathcal{H}$ is a finite dimensionnal vector space. Using Theorem 3.2.2, it turns out that

$$\| \mathbb{P}_{k_0} \dot{u}_k(t)\|_{L^2} \leq \lambda_{k_0} \| \mathbb{P}_{k_0} u_k(t)\|_{\mathcal{V}_\sigma} + \lambda_{k_0} \|u_k(t)\|^{2-\frac{d}{2}}_{L^2} \|\nabla u_k(t)\|^{\frac{d}{2}}_{L^2}.$$

Using energy estimate (4.8), we infer that $(\mathbb{P}_{k_0} \dot{u}_k)_k$ is a bounded sequence of $L^{\frac{4}{d}}([0,T];L^2)$ which means that

$$\forall k\,,\ \| \mathbb{P}_{k_0} \dot{u}_k\|_{L^{\frac{4}{d}}([0,T];L^2)} \leq C_{u_0,f,k_0}.$$

Thus, by integration and Hölder estimate, we get, for any $(t,t') \in [0,T]^2$,

$$\| \mathbb{P}_{k_0} u_k(t) - \mathbb{P}_{k_0} u_k(t)\|_{L^2} \leq |t-t'|^{1\frac{4}{d}} C_{u_0,f,k_0}. \tag{4.12}$$

Moreover, for any t in $[0,T]$, the set $\{P_{k_0} u_k(t),\ k\in\mathbb{N}\}$ in bounded in the finite dimensionnal space $\mathcal{H}_{k_0}$. Thus it is relatively compact. Together with (4.12), this allows to apply Ascoli's compactness theorem. Thus, in particular, the set $\{P_{k_0} u_k,,\ k\in\mathbb{N}\}$ can be recovered by a finite number of balls of radius $\varepsilon/2$. Together with (4.11) , this proves (4.9) and (4.10).

4.6 End of the proof of the Leray Theorem

The local strong convergence of $(u_k)_{k\in\mathbb{N}}$ will be crucial to pass to the limit in $(NS_{\nu,k})$ to obtain solutions of (NS_ν).

According to the definition of a weak solution of (NS_ν), let us consider a test function Ψ in $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$. Because u_k is a solution of $(NS_{\nu,k})$, we have

$$\begin{aligned}\frac{d}{dt}\langle u_k(t),\Psi(t)\rangle &= \langle \dot{u}_k(t),\Psi(t)\rangle + \langle u_k(t),\dot{\Psi}(t)\rangle \\ &= \nu\langle \mathbb{P}_k \Delta u_k(t),\Psi(t)\rangle + \langle \mathbb{P}_k Q(u_k(t),u_k(t)),\Psi(t)\rangle \\ &\qquad + \langle f_k(t),\Psi(t)\rangle + \langle u_k(t),\dot{\Psi}(t)\rangle.\end{aligned}$$

We have after integration by parts

$$\begin{aligned}\langle \mathbb{P}_k \Delta u_k(t),\Psi(t)\rangle &= -\nu(u_k(t)|\,\mathbb{P}_k \Psi(t))_{\mathcal{V}_\sigma} = -\nu(u_k(t)|\Psi(t))_{\mathcal{V}_\sigma} \\ \langle \mathbb{P}_k Q(u_k(t),u_k(t)),\Psi(t)\rangle &= \int_\Omega u_k(t,x)\otimes u_k(t,x) : \nabla\, \mathbb{P}_k \Psi(t,x)\,dx \quad \text{and} \\ \langle u_k(t),\dot{\Psi}(t)\rangle &= \int_\Omega u_k(t,x)\cdot \partial_t \Psi(t,x)\,dx.\end{aligned}$$

By time integration between 0 and t, we infer that

$$\langle u_k(t), \Psi(t)\rangle + \int_0^t \Big(\nu(\nabla u_k(t')|\nabla\Psi(t'))_{\mathcal{V}_\sigma} - (u_k(t')|\partial_t\Psi(t'))_{\mathcal{H}}\Big)dt'$$
$$- \int_0^t\int_\Omega \Big(u_k\otimes u_k : \nabla\,\mathbb{P}_k\,\Psi\Big)dxdt' = \langle u_k(0), \Psi(0)\rangle + \int_0^t \langle f_k(t'), \Psi(t')\rangle\, dt'.$$

We now have to pass to the limit. Using (4.10), we deduce that, for any $t\in[0,T]$,

$$\lim_{k\to\infty} \langle u_k(t), \Psi(t)\rangle = \langle u(t), \Psi(t)\rangle. \tag{4.13}$$

Now, using (4.9) gives

$$\lim_{k\to\infty}\int_0^t (u_k(t')|\partial_t\Psi(t'))_{\mathcal{H}}dt' = \int_0^t (u(t')|\partial_t\Psi(t'))_{\mathcal{H}}dt'. \tag{4.14}$$

Thanks to Theorem 3.2.2, we have

$$\lim_{k\to\infty}\int_0^t \langle f_k(t'), \Psi(t')\rangle\, dt' = \int_0^t \langle f(t'), \Psi(t')\rangle\, dt'. \tag{4.15}$$

Now, we have to treat the non linear term. Let us start by proving the following preliminary lemma.

Lemma 4.6.1 *Let* H *be a Hilbert space, and let* $(A_n)_{n\in\mathbb{N}}$ *be a bounded sequence of linear operators on* H *such that*

$$\forall h\in \mathrm{H},\quad \lim_{n\to\infty}\|A_n h - h\|_{\mathrm{H}} = 0.$$

Then if $\psi\in C([0,T];\mathrm{H})$ *we have* $\displaystyle\lim_{n\to\infty}\sup_{t\in[0,T]}\|A_n\psi(t)-\psi(t)\|_{\mathrm{H}} = 0.$

Proof of Lemma 4.6.1. The function ψ is continuous in time with values in H, so for all positive ε, the compact $\psi([0,T])$, can be covered by a finite number of balls of radius

$$\frac{\varepsilon}{2(\mathcal{A}+1)}\quad\text{with}\quad \mathcal{A}\stackrel{\text{def}}{=}\sup_n\|A_n\|_{\mathcal{L}(\mathrm{H})}.$$

and center $(\psi(t_\ell))_{0\leq\ell\leq N}$. Then we have, for all t in $[0,T]$ and ℓ in $\{0,\cdots,N\}$,

$$\|A_n\psi(t)-\psi(t)\|_{\mathrm{H}} \leq \|A_n\psi(t)-A_n\psi(t_\ell)\|_{\mathrm{H}} + \|A_n\psi(t_\ell)-\psi(t_\ell)\|_{\mathrm{H}} + \|\psi(t_\ell)-\psi(t)\|_{\mathrm{H}}.$$

The assumption on A_n implies that for any ℓ, the sequence $(A_n\psi(t_\ell))_{n\in\mathbb{N}}$ tends to $\psi(t_\ell)$. Thus, an integer n_N exists such that, if $n\geq n_N$,

$$\forall \ell\in\{0,\cdots N\}\,,\ \|A_n\psi(t_\ell)-\psi(t_\ell)\|_{\mathrm{H}} < \frac{\varepsilon}{2}\cdot$$

We infer that, if $n\geq n_N$, for all $t\in[0,T]$ and all $\ell\in\{0,\cdots,N\}$,

$$\|A_n\psi(t)-\psi(t)\|_{\mathrm{H}} \leq \|A_n\psi(t)-A_n\psi(t_\ell)\|_{\mathrm{H}} + \|\psi(t_\ell)-\psi(t)\|_{\mathrm{H}} + \frac{\varepsilon}{2}\cdot$$

For any t, let us choose ℓ such that

$$\|\psi(t)-\psi(t_\ell)\|_{\mathrm{H}} \leq \frac{\varepsilon}{2(\mathcal{A}+1)}\cdot$$

The lemma is proved.

Now let us pass to the limit in the non linear term. As above the sequence u_k is bounded in $L^{\frac{8}{3}}([0,T];L^4(\Omega))$, we have in fact

$$\lim_{k\to\infty}\int_0^t\int_\Omega (u_k\otimes u_k:\nabla\mathbb{P}_k\Psi)(t',x)\,dxdt' = \lim_{k\to\infty}\int_0^t\int_\Omega (u_k\otimes u_k:\nabla\Psi)(t',x)\,dxdt'.$$

So it is enough to prove that

$$\lim_{k\to\infty}\int_0^t\int_\Omega (u_k\otimes u_k:\nabla\Psi)(t',x)\,dxdt' = \int_0^t\int_\Omega (u\otimes u:\nabla\Psi)(t',x)\,dxdt'.$$

It is enough to prove that $\lim_{k\to\infty}\|u_k\otimes u_k - u\otimes u\|_{L^1([0,T];L^2(\Omega))}=0$ which will be implied by

$$\lim_{k\to\infty}\|u_k-u\|_{L^2([0,T];L^4(\Omega))}=0. \tag{4.16}$$

Using (2.2.1), we have

$$\|u_k-u\|_{L^2([0,T];L^4(\Omega))}\leq C\|u_k-u\|_{L^2([0,T]\times\Omega)}^{1-\frac{d}{4}}\|\nabla(u_k-u)\|_{L^2([0,T]\times\Omega)}^{\frac{d}{4}}.$$

Proposition 4.5.1 allows to conclude the proof of the fact that u is a solution of (NS_ν) in the sense of Definition 4.2.1.

It remains to prove the energy inequality (4.5). Assertion (4.10) of Proposition 4.5.1 implies in particular that for any time $t\geq 0$ and any $v\in\mathcal{V}_\sigma$,

$$\lim_{k\to\infty}(u_k(t)|v)_{\mathcal{H}} = \lim_{k\to\infty}\langle u_k(t),v)\rangle = \langle u(t),v)\rangle = (u(t)|v)_{\mathcal{H}}.$$

As $\mathcal{V}_\sigma$ is dense in $\mathcal{H}$, we get that for any $t\geq 0$, the sequence $(u_k(t))_{k\in\mathbb{N}}$ converges weakly towards $u(t)$ in the Hilbert space $\mathcal{H}$. Hence

$$\|u(t)\|_{L^2}^2\leq\liminf_{k\to\infty}\|u_k(t)\|_{L^2}^2\quad\text{for all}\quad t\geq 0.$$

On the other hand, $(u_k)_{k\in\mathbb{N}}$ converges weakly to u in $L^2_{loc}(\mathbb{R}^+;\mathcal{V})$, so that for all non negative t, we have

$$\int_0^t\|\nabla u(t')\|_{L^2}^2\,dt'\leq\liminf_{k\to\infty}\int_0^t\|\nabla u_k(t')\|_{L^2}^2\,dt'.$$

Taking the $\liminf\limits_{k\to\infty}$ in the energy equality for approximate solutions (4.7) yields the energy inequality (4.5).

Remarks

- Ce chapitre is to savoir.
- Si vous êtes curieux, vous pouvez consulter l'article fondateur of J. Leray "Essai on the mouvement of a liquide visqueux emplissant the space, *Acta Matematica*, **63**, 1933, pages 193–248.
- Pour en savoir more on the equation of Navier-Stokes incompressible, vous pouvez consulter by exemple les livres of P. Constantin and C. Foias *Navier-Stokes equations*, Chigago University Press, 1988 et of P.-G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem.* Chapman & Hall/CRC, Research Notes in Mathematics, **431**, 2002.
- Si vous êtes intéressés by des développements liés à la géophysique, vous pouvez consulter the livre of J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Mathematical Geophysics; an introduction to rotating fluids and Navier-Stokes equations*, Oxford Lecture series in Mathematics and its maps, **32**, Oxford University Press, 2006.

Chapter 5

Stability of Navier-Stokes equations

In this chapter we intend to investigate the stability of the Leray solutions constructed in the previous chapter. It is useful to start by analyzing the linearised version of the Navier-Stokes equations, so the first section of the chapter is devoted to the proof of the wellposedness of the time dependent Stokes system. The study will be applied in Section 5.1 to the two dimensional Navier-Stokes equations, and the more delicate case of three space dimensions will be dealt with in Sections 5.2 to 5.3.

5.1 Stability in dimension two

In a two dimensional domain, the Leray weak solutions are unique and even stable. More precisely, we have the following theorem.

Theorem 5.1.1 *For any data u_0 in $\mathcal{H}$ and f in $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$, the Leray weak solution is unique. Moreover, it belongs to $C(\mathbb{R}^+;\mathcal{H})$ and satisfies, for any (s,t) such that $0 \leq s \leq t$,*

$$\frac{1}{2}\|u(t)\|^2_{L^2} + \nu \int_s^t \|\nabla u(t')\|^2_{L^2}\,dt' = \frac{1}{2}\|u(s)\|^2_{L^2} + \int_s^t \langle f(t'), u(t')\rangle\,dt'. \tag{5.1}$$

Furthermore, the Leray solutions are stable in the following sense. Let u (resp. v) be the Leray solution associated with u_0 (resp. v_0) in $\mathcal{H}$ and f (resp. g) in the space $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$ then,

$$\begin{aligned}\|(u-v)(t)\|^2_{L^2} &+ \nu \int_0^t \|\nabla(u-v)(t')\|^2_{L^2}\,dt' \\ &\leq \Big(\|u_0 - v_0\|^2_{L^2} + \frac{1}{\nu}\int_0^t \|(f-g)(t')\|^2_{\mathcal{V}'_\sigma}\,dt'\Big)\exp\Big(\frac{CE^2(t)}{\nu^4}\Big) \quad \textit{with}\end{aligned}$$

$$E(t) \stackrel{\text{def}}{=} \min\Big\{\|u_0\|^2_{L^2} + \frac{1}{\nu}\int_0^t \|f(t')\|^2_{\mathcal{V}'_\sigma}\,dt'\,, \|v_0\|^2_{L^2} + \frac{1}{\nu}\int_0^t \|g(t')\|^2_{\mathcal{V}'_\sigma}\,dt'\Big\}.$$

Proof of Theorem 5.1.1. As u belongs to $L^\infty_{loc}(\mathbb{R}^+;\mathcal{H})\cap L^2_{loc}(\mathbb{R}^+;\mathcal{V}_\sigma)$, thanks to Lemma 4.3.2 page 43, the non linear term $Q(u,u)$ belongs to $L^2_{loc}(\mathbb{R}^+;\mathcal{V}')$. Thus u is the solution of (ES_ν) with initial data u_0 and external force $f + Q(u,u)$. Theorem 4.1.1 immediately implies that u belongs to $C(\mathbb{R}^+;\mathcal{H})$ and satisfies, for any (s,t) such that $0 \leq s \leq t$,

$$\begin{aligned}\frac{1}{2}\|u(t)\|^2_{L^2} + \nu \int_s^t \|\nabla u(t')\|^2_{L^2}\,dt' &= \frac{1}{2}\|u(s)\|^2_{L^2} \\ &+ \int_s^t \langle f(t'), u(t')\rangle\,dt' + \int_s^t \langle Q(u(t'),u(t')), u(t')\rangle\,dt'.\end{aligned}$$

Using Lemma 4.3.2, we get the energy equality (5.1).

To prove the stability, let us observe that, by difference $w \stackrel{\text{def}}{=} u-v$ is the solution of (ES_ν) with data $u_0 - v_0$ and external force $f - g + Q(u,u) - Q(v,v)$, Theorem 4.1.1 implies that

$$
\begin{aligned}
\|w(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla w(t')\|_{L^2}^2\, dt' &= \|w(0)\|_{L^2}^2 \\
&+ 2\int_0^t \langle (f-g)(t'), w(t')\rangle\, dt' + 2\int_0^t \langle (Q(u,u) - Q(v,v))(t'), w(t')\rangle\, dt'.
\end{aligned}
$$

The non linear term is estimated thanks to the following lemma.

Lemma 5.1.1 *In two dimensional domains, if a and b belong to $\mathcal{V}_\sigma$, we have*

$$
|\langle (Q(a,a) - Q(b,b)), a-b\rangle| \leq C\|\nabla(a-b)\|_{L^2}^{\frac{3}{2}} \|a-b\|_{L^2}^{\frac{1}{2}} \|\nabla a\|_{L^2}^{\frac{1}{2}} \|a\|_{L^2}^{\frac{1}{2}}.
$$

Proof of Lemma 5.1.1. It is an exercise of algebra to deduce from Lemma 4.3.2 that

$$
\langle Q(a,a) - Q(b,b), a-b\rangle = \langle Q(a-b,a), a-b\rangle. \tag{5.2}
$$

Using again Lemma 4.3.2, we get the result.

Let us go back to the proof of Theorem 5.1.1. Using that $2ab \leq a^2 + b^2$, we get

$$
\begin{aligned}
\|w(t)\|_{L^2}^2 + \frac{3}{2}\nu \int_0^t \|\nabla w(t')\|_{L^2}^2\, dt' &\leq \|w(0)\|_{L^2}^2 + \frac{2}{\nu}\int_0^t \|(f-g)(t')\|_{\mathcal{V}'_\sigma}^2\, dt' \\
&+ C\int_0^t \|\nabla w(t')\|_{L^2}^{\frac{3}{2}} \|w(t')\|_{L^2}^{\frac{1}{2}} \|\nabla u(t')\|_{L^2}^{\frac{1}{2}} \|u(t')\|_{L^2}^{\frac{1}{2}}\, dt'.
\end{aligned}
$$

Using (with $\theta = 1/4$) the convexity inequality

$$
ab \leq \theta a^{\frac{1}{\theta}} + (1-\theta) b^{1-\frac{1}{\theta}} \tag{5.3}
$$

we infer that

$$
\begin{aligned}
\|w(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla w(t')\|_{L^2}^2\, dt' &\leq \|w(0)\|_{L^2}^2 + \frac{2}{\nu}\int_0^t \|(f-g)(t')\|_{\mathcal{V}'_\sigma}^2\, dt' \\
&+ \frac{C}{\nu^3}\int_0^t \|w(t')\|_{L^2}^2 \left(\|\nabla u(t')\|_{L^2}^2 \|u(t')\|_{L^2}^2\right)\, dt'.
\end{aligned}
$$

Gronwall's lemma implies that

$$
\begin{aligned}
\|w(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla w(t')\|_{L^2}^2\, dt' \quad &\leq \quad \Big(\|w(0)\|_{L^2}^2 + \frac{2}{\nu}\int_0^t \|(f-g)(t')\|_{\mathcal{V}'_\sigma}^2\, dt'\Big) \\
&\times \exp\biggl(\frac{C}{\nu^3} \sup_{t'\in[0,t]} \|u(t')\|_{L^2}^2 \int_0^t \|\nabla u(t')\|_{L^2}^2\, dt'\biggr).
\end{aligned}
$$

The energy estimate tells us that

$$
\sup_{t'\in[0,t]} \|u(t')\|_{L^2}^2 \int_0^t \|\nabla u(t')\|_{L^2}^2\, dt' \leq \frac{1}{\nu}\biggl(\|u_0\|_{L^2}^2 + \frac{2}{\nu}\int_0^t \|f(t')\|_{\mathcal{V}'_\sigma}^2\, dt'\biggr)^2.
$$

As u and v play the same role, the theorem is proved.

5.2 Stability in dimension three

In order to get stability, we need to enforce the time regularity of the Leray solution. The precise stability theorem is the following.

Theorem 5.2.1 *Let u be a Leray solution associated with initial velocity u_0 in $\mathcal{H}$ and bulk force f in $L^2([0,T];\mathcal{V}')$. We assume that u belongs to the space $L^4([0,T];\mathcal{V}_\sigma)$ for some positive T. Then u is unique, belongs to $C([0,T];\mathcal{H})$ and satisfies, for any (s,t) such that $0 \leq s \leq t \leq T$,*

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \nu\int_s^t \|\nabla u(t')\|_{L^2}^2\,dt' = \frac{1}{2}\|u(s)\|_{L^2}^2 + \int_s^t \langle f(t'),u(t')\rangle\,dt'. \tag{5.1}$$

Let v be any solution associated with v_0 in $\mathcal{H}$ and g in $L^2_{loc}([0,T];\mathcal{V}')$. Then, for all t in $[0,T]$,

$$\begin{aligned}&\|(u-v)(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla(u-v)(t')\|_{L^2}^2\,dt' \\ &\qquad\leq \left(\|u_0-v_0\|_{L^2}^2 + \frac{2}{\nu}\int_0^t \|(f-g)(t')\|_{\mathcal{V}'_\sigma}^2\,dt'\right)\exp\Big(\frac{C}{\nu^3}\int_0^t \|\nabla u(t')\|_{L^2}^4\,dt'\Big).\end{aligned}$$

The proof that such a $L^4([0,T];\mathcal{V}_\sigma)$ solution u exists will be detailed in Section 5.3 in the case of bounded domains .

Proof of Theorem 5.2.1. Thanks to Lemma 4.3.2, the fact that u belongs to $L^4([0,T];\mathcal{V}_\sigma)$ implies that

$$\begin{aligned}\|Q(u,u)\|_{L^2([0,T];\mathcal{V}')} &\leq C\|u\|_{L^\infty([0,T];L^2)}^{\frac{1}{2}}\|u\|_{L^3([0,T];H_0^1)}^{\frac{3}{2}} \\ &\leq CT^{\frac{1}{8}}\|u\|_{L^\infty([0,T];L^2)}^{\frac{1}{2}}\|u\|_{L^4([0,T];H_0^1)}^{\frac{3}{2}}.\end{aligned} \tag{5.2}$$

Hence the non linear term $Q(u,u)$ belongs to $L^2([0,T];\mathcal{V}')$. Thus, exactly as in the two dimensional case, u is the solution of (ES_ν) with initial data u_0 and external force $f+Q(u,u)$. Theorem 4.1.1 immediately implies that u belongs to $C([0,T];\mathcal{H})$ and satisfies, for any (s,t) such that $0 \leq s \leq t$,

$$\begin{aligned}\frac{1}{2}\|u(t)\|_{L^2}^2 + \nu\int_s^t \|\nabla u(t')\|_{L^2}^2\,dt' &= \frac{1}{2}\|u(s)\|_{L^2}^2 \\ &\quad + \int_s^t \langle f(t'),u(t')\rangle\,dt' + \int_s^t \langle Q(u(t'),u(t')),u(t')\rangle\,dt'.\end{aligned}$$

Using Lemma 4.3.2, we get the energy equality (5.1).

As u and v are two Leray solutions, we can write that

$$\begin{aligned}\delta_\nu(t) &\stackrel{\text{def}}{=} \|(u-v)(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla(u-v)(t')\|_{L^2}^2\,dt' \\ &= \|u(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla u(t')\|_{L^2}^2 dt' + \|v(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla v(t')\|_{L^2}^2\,dt' \\ &\qquad - 2(u(t)|v(t))_{L^2} - 4\nu\int_0^t (\nabla u(t')|\nabla v(t'))_{L^2}\,dt' \\ &\leq \|u_0\|_{L^2}^2 + 2\int_0^t \langle f(t'),u(t')\rangle\,dt' + \|v_0\|_{L^2}^2 + 2\int_0^t \langle g(t'),v(t')\rangle\,dt' \\ &\qquad - 2(u(t)|v(t))_{L^2} - 4\nu\int_0^t (\nabla u(t')|\nabla v(t'))_{L^2}\,dt'.\end{aligned} \tag{5.3}$$

Now the problem consists in evaluating the cross-product terms

$$(u(t)|v(t))_{L^2} + 2\nu \int_0^t (\nabla u(t')|\nabla v(t'))_{L^2}\, dt' - \int_0^t \langle g(t'), v(t')\rangle\, dt' - \int_0^t \langle f(t'), u(t')\rangle\, dt'.$$

Let us suppose a moment that u and v are smooth in space and time; then we can simply multiply scalarly by v the equation satisfied by u, and conversely multiply scalarly by u the equation satisfied by v. We get

$$\begin{aligned}(\partial_t u|v)_{L^2} + \nu(\nabla u|\nabla v)_{L^2} + (u\cdot\nabla u|v)_{L^2} &= \langle f, v\rangle \quad \text{and}\\ (\partial_t v|u)_{L^2} + \nu(\nabla v|\nabla u)_{L^2} + (v\cdot\nabla v|u)_{L^2} &= \langle g, u\rangle\end{aligned}$$

hence summing both equalities yields

$$\begin{aligned}\partial_t(u(t)|v(t))_{L^2} + 2\nu(\nabla u(t)|\nabla v(t))_{L^2} - \langle f(t), v(t)\rangle - \langle g(t)|u(t)\rangle&\\ + (u(t)\cdot\nabla u(t)|v(t))_{L^2} + (v(t)\cdot\nabla v(t)|u(t))_{L^2} &= 0.\end{aligned}$$

After an easy algebraic computation we find that

$$(u(t)\cdot\nabla u(t)|v(t))_{L^2} + (v(t)\cdot\nabla v(t)|u(t))_{L^2} = ((u-v)(t)\cdot\nabla(u-v)(t)|u(t))_{L^2},$$

hence after integration in time, we obtain

$$\begin{aligned}(u(t)|v(t))_{L^2} + 2\nu\int_0^t (\nabla u(t')|\nabla v(t'))^2_{L^2}\, dt' - \int_0^t \langle g(t'), v(t')\rangle\, dt' - \int_0^t \langle f(t'), u(t')\rangle\, dt'&\\ = -(u_0|v_0)_{L^2} + \int_0^t \Big((u-v)(t')\cdot\nabla(u-v)(t')|u(t')\Big)_{L^2} dt'&\\ + \int_0^t \langle f(t'), (v-u)(t')\rangle\, dt' + \int_0^t \langle g(t'), (u-v)(t')\rangle\, dt'.& \quad (5.4)\end{aligned}$$

Plugging (5.4) into (5.3) yields

$$\begin{aligned}\|(u-v)(t)\|^2_{L^2} + 2\nu\int_0^t \|\nabla(u-v)(t')\|^2_{L^2}\, dt' \leq \|u_0 - v_0\|^2_{L^2}&\\ + 2\int_0^t \langle f-g, u-v\rangle(t')dt' + 2\int_0^t \Big((u-v)\cdot\nabla(u-v)|u\Big)_{L^2}(t')dt',&\end{aligned}$$

and a Gronwall lemma gives the smallness of δ_ν. However unfortunately the above computations make no sense if no smoothness in space and time is known on u or on v (and in particular the final Gronwall argument does not seem possible to write correctly). So some precautions have to be taken in order to make those computations valid, and to conclude the argument by a Gronwall lemma — in particular we are going to see that the assumption that $u \in L^4([0,T];\mathcal{V}_\sigma)$ is enough to make the above computations valid.

Let us therefore proceed with the rigorous computations. Identity (5.3) involves scalar products of u and v, which naturally leads to using the definition of weak solutions choosing for instance u as a test function. Unfortunately, in order to be admissible, test functions need to belong to $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$, so that some preliminary smoothing in time of u is required. We shall use the following approximation lemma and postpone its proof to the end of the proof of the theorem.

Lemma 5.2.1 *Let u be a Leray solution which belongs to $L^4([0,T];\mathcal{V}_\sigma)$. A sequence $(\widetilde{u}_k)_{k\in\mathbb{N}}$ of $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$ exists such that*

- *the sequence* $(\widetilde{u}_k)_{k\in\mathbb{N}}$ *tends to* u *in* $L^4([0,T];\mathcal{V}_\sigma)\cap L^\infty([0,T];\mathcal{H})$*;*
- *for all* $k\in\mathbb{N}$*, we have*

$$\partial_t\widetilde{u}_k-\nu\Delta\widetilde{u}_k=Q(\widetilde{u}_k,\widetilde{u}_k)+f+R_k+\nabla p_k \quad \text{with} \quad \lim_{k\to\infty}\|R_k\|_{L^2([0,T];\mathcal{V}'_\sigma)}=0. \tag{5.5}$$

The function $\widetilde{u}_k$ belongs to $C^1(\mathbb{R}^+;\mathcal{V}_\sigma)$, thus it can be used as a test function in Definition 4.2.1. As v is a Leray solution, we have

$$\begin{aligned}\mathcal{B}_k(t) &\stackrel{\text{def}}{=} (v(t)|\widetilde{u}_k(t))_{L^2}\\ &= (v(0)|\widetilde{u}_k(0))_{L^2}-\nu\int_0^t(\nabla v(t')|\nabla\widetilde{u}_k(t'))_{L^2}dt'+\int_0^t\langle g(t'),\widetilde{u}_k(t')\rangle\,dt'\\ &\qquad+\int_0^t(v(t')\otimes v(t')|\nabla\widetilde{u}_k(t'))_{L^2}\,dt'+\int_0^t\langle v(t'),\partial_t\widetilde{u}_k(t')\rangle\,dt'.\end{aligned}$$

Thanks to (5.5), we get

$$\begin{aligned}\mathcal{B}_k(t) &= (v(0)|\widetilde{u}_k(0))_{L^2}-2\nu\int_0^t(\nabla v(t')|\nabla\widetilde{u}_k(t'))_{L^2}dt'+\int_0^t\langle g(t'),\widetilde{u}_k(t')\rangle dt'\\ &\qquad+\int_0^t\langle f(t'),v(t')\rangle\,dt'+\int_0^t(v(t')\otimes v(t')|\nabla\widetilde{u}_k(t'))_{L^2}\,dt'\\ &\qquad+\int_0^t\langle Q(\widetilde{u}_k(t'),\widetilde{u}_k(t')),v(t')\rangle\,dt'+\int_0^t\langle v(t'),R_k(t')\rangle\,dt'.\end{aligned}$$

Lemma 5.2.1 implies that $\lim_{k\to\infty}\mathcal{B}_k(t)=(v(t)|u(t))_{L^2}$ and that

$$\begin{aligned}\lim_{k\to\infty}\Big\{(v(0)|\widetilde{u}_k(0))_{L^2}-2\nu\int_0^t(\nabla v(t')|\nabla\widetilde{u}_k(t'))_{L^2}\,dt'\\ +\int_0^t\langle v(t'),R_k(t')\rangle\,dt'+\int_0^t\langle g(t'),\widetilde{u}_k(t')\rangle\,dt'\Big\}\end{aligned}$$

is equal to $(v(0)|u(0))_{L^2}-2\nu\int_0^t(\nabla v(t')|\nabla u(t'))_{L^2}\,dt'+\int_0^t\langle g(t'),u(t')\rangle\,dt'$. Thus defining

$$\mathcal{N}_k(t)\stackrel{\text{def}}{=}\int_0^t(v(t')\otimes v(t')|\nabla\widetilde{u}_k(t'))_{L^2}\,dt'+\int_0^t\langle Q(\widetilde{u}_k(t'),\widetilde{u}_k(t')),v(t')\rangle\,dt',$$

we obtain

$$\begin{aligned}(v(t)|u(t))_{L^2}=(v(0)|u(0))_{L^2}-2\nu\int_0^t(\nabla v(t')|\nabla u(t'))_{L^2}\,dt'\\ +\int_0^t\langle g(t'),u(t')\rangle\,dt'+\int_0^t\langle f(t'),v(t')\rangle\,dt'+\lim_{k\to\infty}\mathcal{N}_k(t).\end{aligned}$$

Plugging this into (5.3) gives

$$\delta_\nu(t)=\|u_0-v_0\|^2_{L^2}+2\int_0^t\langle(f-g)(t'),(u-v)(t')\rangle\,dt'+\lim_{k\to\infty}\mathcal{N}_k(t).$$

It remains to study the term $\mathcal{N}_k(t)$. In order to do this, let us observe that, for any vector field a and b in $\mathcal{V}_\sigma$, we have $(b\otimes b|\nabla a)_{L^2} = \langle Q(b,b),a\rangle$ and thus

$$(b\otimes b|\nabla a)_{L^2} + \langle Q(a,a),b\rangle = \langle Q(b,b),a\rangle + \langle Q(a,a),b\rangle.$$

Using Lemma 4.3.2, we can write

$$\begin{aligned}\langle Q(b,b),a\rangle + \langle Q(a,a),b\rangle &= \langle Q(b,b),a-b\rangle + \langle Q(a,a),b-a\rangle\\ &= \langle Q(b,a),a-b\rangle + \langle Q(a,a),b-a\rangle.\end{aligned}$$

Thus, it turns out that

$$\begin{aligned}\langle Q(b,b),a\rangle + \langle Q(a,a),b\rangle &= \langle Q(a-b,a),b-a\rangle\\ &= ((a-b)\otimes a|\nabla(b-a))_{L^2}.\end{aligned}$$

Using the Gagliardo–Nirenberg inequality (see Corollary 2.2.1), we get for any a and c in $\mathcal{V}_\sigma$,

$$|(c\otimes a|\nabla c)_{L^2}| \leq C\|a\|_{L^6}\|c\|_{L^3}\|\nabla c\|_{L^2} \leq C\|\nabla a\|_{L^2}\|c\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{L^2}^{\frac{3}{2}}. \tag{5.6}$$

For almost every time t, the vector field $v(t)$ belongs to $\mathcal{V}_\sigma$. It follows that for all $k\in\mathbb{N}$ and $t\geq 0$, taking $a=\widetilde{u}_k(t')$ and $b=v(t')$, $t'\in[0,t]$, we have

$$\mathcal{N}_k(t)\leq C\int_0^t \|\nabla\widetilde{u}_k(t')\|_{L^2}\|(\widetilde{u}_k-v)(t')\|_{L^2}^{\frac{1}{2}}\|\nabla(\widetilde{u}_k-v)(t')\|_{L^2}^{\frac{3}{2}}\,dt'.$$

Using Lemma 5.2.1, we know that

$$\begin{array}{rcl}(\|\nabla\widetilde{u}_k(\cdot)\|_{L^2})_{k\in\mathbb{N}} & \text{tends to} & \|\nabla u(\cdot)\|_{L^2} \quad\text{in}\quad L^4([0,T])\\ (\|(\widetilde{u}_k-v)(t')\|_{L^2})_{k\in\mathbb{N}} & \text{tends to} & \|(u-v)(\cdot)\|_{L^2} \quad\text{in}\quad L^\infty([0,T])\\ (\|\nabla(\widetilde{u}_k-v)(\cdot)\|_{L^2})_{k\in\mathbb{N}} & \text{tends to} & \|\nabla(u-v)(\cdot)\|_{L^2} \quad\text{in}\quad L^2([0,T]).\end{array}$$

Therefore, we have

$$\lim_{k\to\infty}\mathcal{N}_k(t)\leq \int_0^t \|\nabla u(t')\|_{L^2}\|(u-v)(t')\|_{L^2}^{\frac{1}{2}}\|\nabla(u-v)(t')\|_{L^2}^{\frac{3}{2}}\,dt'.$$

We conclude that

$$\begin{aligned}\delta_\nu(t)\leq \|u_0-v_0\|_{L^2}^2 &+ 2\int_0^t \|(f-g)(t')\|_{\mathcal{V}'_\sigma}\|(u-v)(t')\|_{\mathcal{V}}\,dt'\\ &+ C\int_0^t \|\nabla u(t')\|_{L^2}\|\nabla(u-v)(t')\|_{L^2}^{\frac{3}{2}}\|(u-v)(t')\|_{L^2}^{\frac{1}{2}}\,dt'.\end{aligned}$$

Using the convexity inequality (5.3) with $\theta=1/4$ and $\theta=1/2$, we obtain

$$\begin{aligned}\|(u-v)(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla(u-v)(t')\|_{L^2}^2 dt' &\leq \|u_0-v_0\|_{L^2}^2\\ &+ \frac{2}{\nu}\int_0^t \|(f-g)(t')\|_{\mathcal{V}'_\sigma}^2\,dt' + \int_0^t \frac{C}{\nu^3}\|\nabla u(t')\|_{L^2}^4\|(u-v)(t')\|_{L^2}^2\,dt'.\end{aligned}$$

Gronwall's Lemma allows to conclude the proof of Theorem 5.2.1 provided we prove Lemma 5.2.1.

Proof of Lemma 5.2.1. Thanks to Inequality (5.2), $Q(u,u)$ belongs to $L^2([0,T];\mathcal{V}')$. This implies that if in addition u is a Leray solution, then $\partial_t u$ also belongs to $L^2([0,T];\mathcal{V}')$. Lebesgue's Theorem together with Theorem 3.2.2 page 30 yields

$$\lim_{k\to\infty} \|\,\mathbb{P}_k\, u - u\|_{L^4([0,T];\mathcal{V}_\sigma)} = \lim_{k\to\infty} \|\,\mathbb{P}_k\, \partial_t u - \partial_t u\|_{L^2([0,T];\mathcal{V}'_\sigma)} = 0.$$

Then as in Lemma 4.3.1 page 43, we can define by a standard regularization procedure in time, a sequence $(\widetilde{u}_k)_{k\in\mathbb{N}}$ in $C^1(\mathbb{R}^+,\mathcal{V}_\sigma)$ such that $\widetilde{u}_k$ tends to u in $L^4([0,T];\mathcal{V}_\sigma)\cap L^\infty([0,T],\mathcal{H})$. Moreover $\partial_t\widetilde{u}_k$ tends to $\partial_t u$ in $L^2([0,T];\mathcal{V}'_\sigma)$ and Inequality (5.2) implies that

$$\lim_{k\to\infty} \|Q(\widetilde{u}_k,\widetilde{u}_k) - Q(u,u)\|_{L^2([0,T];\mathcal{V}'_\sigma)} = 0.$$

Lemma 5.2.1 and thus Theorem 5.2.1 are now proved.

5.3 Stable solutions in a bounded domain

The purpose of this section is the proof of the existence of solutions of the system (NS_ν) which are L^4 in time with values in $\mathcal{V}_\sigma$. In order to state (and prove) a sharp theorem, we shall introduce intermediate spaces between the spaces $\mathcal{V}'_\sigma$ and $\mathcal{V}_\sigma$. Then, we shall prove a global existence theorem for small data and then a local in time theorem for large data.

5.3.1 Intermediate spaces

We shall define a family of intermediate spaces between the spaces $\mathcal{V}'_\sigma$ and $\mathcal{V}_\sigma$. This can be done by abstract interpolation theory but we prefer to do it here in an explicit way.

Definition 5.3.1 *Let s be in $[-1,1]$. We shall denote by $\mathcal{V}^s_\sigma$ the space of vector fields u in $\mathcal{V}'$ such that*

$$\|u\|^2_{\mathcal{V}^s_\sigma} \stackrel{\text{def}}{=} \sum_{j\in\mathbb{N}} \mu_j^{2s}\langle u, e_j\rangle^2 < +\infty.$$

Theorem 3.2.2 implies that $\mathcal{V}^0_\sigma = \mathcal{H}$ and $\mathcal{V}^1_\sigma = \mathcal{V}_\sigma$. Moreover, it is obvious that, when s is non negative, $\mathcal{V}^s_\sigma$ endowed with the norm $\|\cdot\|_{\mathcal{V}^s_\sigma}$ is a Hilbert space.

The following proposition will be important in the following two paragraphs.

Proposition 5.3.1 *The space $\mathcal{V}^{\frac12}_\sigma$ is embedded in L^3 and the space $L^{\frac32}$ is embedded in $\mathcal{V}^{-\frac12}_\sigma$.*

Proof of Proposition 5.3.1. This proposition can be proved using abstract interpolation theory. We prefer to present here a self contained proof in the spirit of the proof of Theorem 2.2.1. Let us consider a in $\mathcal{V}^{\frac12}_\sigma$. Without any loss of generality, we can assume that $\|a\|_{\mathcal{V}^{\frac12}_\sigma} \leq 1$. Let us define, for a positive real number Λ,

$$a_\Lambda \stackrel{\text{def}}{=} \sum_{j\,/\,\mu_j<\Lambda} \langle a, e_j\rangle e_j \quad \text{and} \quad b_\Lambda \stackrel{\text{def}}{=} a - a_\Lambda.$$

Using the fact that $\{x \in \Omega / |a(x)| > \Lambda\} \subset \{x \in \Omega / |a_\Lambda(x)| > \Lambda/2\} \cup \{x \in \Omega / |b_\Lambda(x)| > \Lambda/2\}$, we can write

$$\begin{aligned}
\|a\|_{L^3}^3 &\leq 3\int_0^{+\infty} \Lambda^2 \text{meas}\big(\{x \in \Omega \,/\, |a_\Lambda(x)| > \Lambda/2\}\big)\, d\Lambda \\
&\qquad\qquad + 3\int_0^{+\infty} \Lambda^2 \text{meas}\big(\{x \in \Omega \,/\, |b_\Lambda(x)| > \Lambda/2\}\big)\, d\Lambda \\
&\leq 3 \times 2^6 \int_0^{+\infty} \Lambda^{-4} \|a_\Lambda\|_{L^6}^6 \, d\Lambda + 3 \times 2^2 \int_0^{+\infty} \|b_\Lambda\|_{L^2}^2 \, d\Lambda.
\end{aligned}$$

Thanks to Theorem 2.2.1, we have, by definition of the $\|\cdot\|_{\mathcal{V}_\sigma^s}$ norm,

$$\begin{aligned}
\|a_\Lambda\|_{L^6}^2 &\leq C\|a_\Lambda\|_{\mathcal{V}_\sigma}^2 \\
&\leq C \sum_{j \,/\, \mu_j < \Lambda} \mu_j^2 \langle a, e_j\rangle^2 \\
&\leq C\Lambda \sum_{j \,/\, \mu_j < \Lambda} \mu_j \langle a, e_j\rangle^2 \leq C\Lambda.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|a\|_{L^3}^3 &\leq C\int_0^{+\infty} \Lambda^{-2} \|a_\Lambda\|_{\mathcal{V}_\sigma}^2 \, d\Lambda + C\int_0^{+\infty} \|b_\Lambda\|_{L^2}^2 \, d\Lambda \\
&\leq C\sum_{j\in\mathbb{N}} \int_{\mu_j}^{+\infty} \Lambda^{-2}\mu_j^2 \langle a, e_j\rangle^2 \, d\Lambda + C\sum_{j\in\mathbb{N}} \int_0^{\mu_j} \langle a, e_j\rangle^2 \, d\Lambda \\
&\leq C\sum_{j\in\mathbb{N}} \mu_j \langle a, e_j\rangle^2 \\
&\leq C.
\end{aligned}$$

This proves the first part of the proposition.

The second part is obtained by a duality argument. By definition, we have, for any a in $\mathcal{V}'$,

$$\begin{aligned}
\|a\|_{\mathcal{V}_\sigma^{-\frac{1}{2}}} &= \|(\mu_j^{-\frac{1}{2}} \langle a, e_j\rangle)_{j\in\mathbb{N}}\|_{\ell^2} \\
&= \sup_{\substack{(\alpha_j)_{j\in\mathbb{N}} \\ \|(\alpha_j)_{j\in\mathbb{N}}\|_{\ell^2} \leq 1}} \sum_{j\in\mathbb{N}} \alpha_j \mu_j^{-\frac{1}{2}} \langle a, e_j\rangle. \qquad (5.1)
\end{aligned}$$

The map L defined by

$$L \left\{ \begin{array}{ccc} \ell^2 & \to & \mathcal{V}_\sigma^{\frac{1}{2}} \\ (\alpha_j)_{j\in\mathbb{N}} & \mapsto & \displaystyle\sum_{j\in\mathbb{N}} \alpha_j \mu_j^{-\frac{1}{2}} e_j \end{array} \right.$$

is an onto isometry. Thus, thanks to (5.1), we have

$$\|a\|_{\mathcal{V}_\sigma^{-\frac{1}{2}}} = \sup_{\|\varphi\|_{\mathcal{V}_\sigma^{\frac{1}{2}}} \leq 1} \sum_{j\in\mathbb{N}} (L^{-1}\varphi)_j \mu_j^{-\frac{1}{2}} \langle a, e_j\rangle.$$

For any φ in $\mathcal{V}_\sigma$, we have

$$\sum_{j\in\mathbb{N}}(L^{-1}\varphi)_j\mu_j^{-\frac{1}{2}}\langle a,e_j\rangle = \langle a,\varphi\rangle.$$

If we assume that a is in $L^{\frac{3}{2}}$, we have, because φ is in L^3,

$$\langle a,\varphi\rangle = \int_\Omega a(x)\cdot\varphi(x)\,dx.$$

Hölder's inequality and the first part of Proposition 5.3.1 imply that

$$|\langle a,\varphi\rangle| \leq \|a\|_{L^{\frac{3}{2}}}\|\varphi\|_{L^3} \leq C\|a\|_{L^{\frac{3}{2}}}\|\varphi\|_{\mathcal{V}_\sigma^{\frac{1}{2}}}.$$

Thus we have

$$\|a\|_{\mathcal{V}_\sigma^{-\frac{1}{2}}} \leq \sup_{\|\varphi\|_{\mathcal{V}_\sigma^{\frac{1}{2}}}\leq 1}\langle a,\varphi\rangle \leq C\|a\|_{L^{\frac{3}{2}}}.$$

This completes the proof of Proposition 5.3.1.

5.3.2 The wellposedness result

The aim of this paragraph is the proof of the following existence theorem with data in $\mathcal{V}_\sigma^{\frac{1}{2}}$.

Theorem 5.3.1 *If the initial data u_0 belongs to $\mathcal{V}_\sigma^{\frac{1}{2}}$ and the bulk force f belongs to the space $L^2_{loc}(\mathbb{R}_+;\mathcal{V}_\sigma^{-\frac{1}{2}})$, then a positive time T exists such that a solution u of (NS_ν) exists in $L^4([0,T];\mathcal{V}_\sigma)$. This solution is unique and belongs to $C([0,T];\mathcal{V}_\sigma^{\frac{1}{2}})$.*

Moreover, a constant c exists (which can be chosen independent of the domain Ω) such that, if

$$\|u_0\|_{\mathcal{V}_\sigma^{\frac{1}{2}}} + \frac{1}{\nu}\|f\|_{L^2(\mathbb{R}^+;\mathcal{V}_\sigma^{-\frac{1}{2}})} \leq c\nu,$$

then the above solution is global.

Proof of Theorem 5.3.1. For the sake of ifmplicity, we shall ignore the bulk force in the proof. Let us consider the sequence $(u_k)_{k\in\mathbb{N}}$ used in the proof of Leray's theorem and defined by the ordinary differential equation $(NS_{\nu,k})$ page 44. The point is to prove that this sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $L^4([0,T];\mathcal{V}_\sigma)$ for some positive T.

Let us recall the remark page 41 which tells us that

$$\begin{aligned} u_k &= \sum_{j=0}^{k} U_{j,k}(t)e_j \quad\text{with}\\ U_{j,k}(t) &\stackrel{\text{def}}{=} (u_0|e_j)_{L^2}e^{-\nu\mu_j^2 t} + \int_0^t e^{-\nu\mu_j^2(t-t')}\langle Q(u_k(t'),u_k(t')),e_j\rangle\,dt'. \end{aligned} \tag{5.2}$$

Using Proposition 5.3.1 we claim that for any vector field a and b in $\mathcal{V}_\sigma$ and for all $j\in\mathbb{N}$,

$$\begin{aligned} \|\,\mathbb{P}_j\operatorname{div}(a\otimes b)\|_{\mathcal{V}_\sigma^{-\frac{1}{2}}} &= \|\,\mathbb{P}_j(a\cdot\nabla b)\|_{\mathcal{V}_\sigma^{-\frac{1}{2}}}\\ &\leq C\|a\cdot\nabla b\|_{L^{\frac{3}{2}}}\\ &\leq C\|a\|_{L^6}\|\nabla b\|_{L^2}. \end{aligned}$$

Using Sobolev embeddings, we deduce that

$$\| \mathbb{P}_j \operatorname{div}(a \otimes b)\|_{\mathcal{V}_\sigma^{-\frac12}} \leq C\|a\|_{\mathcal{V}_\sigma}\|b\|_{\mathcal{V}_\sigma}. \tag{5.3}$$

By definition of the norm on $\mathcal{V}_\sigma^{-\frac12}$, we infer that for all $k \in \mathbb{N}$, a sequence $(c_{j,k}(t))_{j\in\mathbb{N}}$ exists such that

$$|\langle Q(u_k(t), u_k(t)), e_j\rangle| \leq Cc_{j,k}(t)\mu_j^{\frac12}\|u_k(t)\|^2_{\mathcal{V}_\sigma} \tag{5.4}$$

with, for any t, $\displaystyle\sum_{j\in\mathbb{N}} c^2_{j,k}(t) = 1$. Plugging this inequality into (5.2), we get

$$|U_{j,k}(t)| \leq |(u_0|e_j)|e^{-\nu\mu_j^2 t} + C\mu_j^{\frac12}\int_0^t e^{-\nu\mu_j^2(t-t')}c_{j,k}(t')\|u_k(t')\|^2_{\mathcal{V}_\sigma}dt'. \tag{5.5}$$

Thanks to Young's inequality $\|f \star g\|_{L^4} \leq \|f\|_{L^{\frac43}}\|g\|_{L^2}$, we have, for any positive T,

$$\|U_{j,k}\|_{L^4([0,T])} \leq |(u_0|e_j)|\mu_j^{-\frac12}\Big(\frac{1-e^{-4\nu\mu_j^2 T}}{4\nu}\Big)^{\frac14} + \frac{C}{\nu^{\frac34}}\mu_j^{-1}\left(\int_0^T c^2_{j,k}(t)\|u_k(t)\|^4_{\mathcal{V}_\sigma}\,dt\right)^{\frac12}.$$

Multiplying by μ_j and taking the ℓ^2 norm gives

$$\Big(\sum_{j\in\mathbb{N}}\mu_j^2\|U_{j,k}\|^2_{L^4([0,T])}\Big)^{\frac12} \leq \Big(\sum_{j\in\mathbb{N}}\mu_j(u_0|e_j)^2\Big(\frac{1-e^{-4\nu\mu_j^2T}}{\nu}\Big)^{\frac12}\Big)^{\frac12}$$
$$+\frac{C}{\nu^{\frac34}}\Big(\sum_{j\in\mathbb{N}}\int_0^T c^2_{j,k}(t)\|u_k(t)\|^4_{\mathcal{V}_\sigma}dt\Big)^{\frac12}.$$

Thanks to (5.4), we have

$$\Big(\sum_{j\in\mathbb{N}}\mu_j^2\|U_{j,k}\|^2_{L^4([0,T])}\Big)^{\frac12} \leq \Big(\sum_{j\in\mathbb{N}}\mu_j(u_0|e_j)^2\Big(\frac{1-e^{-4\nu\mu_j^2T}}{\nu}\Big)^{\frac12}\Big)^{\frac12} + \frac{C}{\nu^{\frac34}}\|u_k\|^2_{L^4([0,T];\mathcal{V}_\sigma)}.$$

Now let us observe that, thanks to the Cauchy–Schwarz inequality, for any a in $\ell^2(L^4[0,T])$,

$$\begin{aligned}
\int_0^T \|a_j(t)\|^4_{\ell^2(\mathbb{N})}dt &= \int_0^T\Big(\sum_{j\in\mathbb{N}}a_j^2(t)\Big)^2dt\\
&= \sum_{j\in\mathbb{N},k\in\mathbb{N}}\int_0^T a_j^2(t)a_k^2(t)dt\\
&\leq \sum_{j\in\mathbb{N},k\in\mathbb{N}}\|a_j\|^2_{L^4([0,T])}\|a_k\|^2_{L^4([0,T])}\\
&\leq \Big\|(\|a_j\|_{L^4([0,T])})_{j\in\mathbb{N}}\Big\|^4_{\ell^2}
\end{aligned}$$

Let us notice that this is a particular case of the Minkowski inequality. Thus we infer that

$$\|u_k\|_{L^4([0,T];\mathcal{V}_\sigma)} \leq \Big(\sum_{j\in\mathbb{N}}\mu_j(u_0|e_j)^2\Big(\frac{1-e^{-4\nu\mu_j^2T}}{\nu}\Big)^{\frac12}\Big)^{\frac12} + \frac{C}{\nu^{\frac34}}\|u_k\|^2_{L^4([0,T];\mathcal{V}_\sigma)}.$$

Let us define $T_k \overset{\text{def}}{=} \sup\Big\{T>0\,/\,\|u_k\|_{L^4([0,T];\mathcal{V}_\sigma)} \leq \dfrac{\nu^{\frac34}}{2C}\Big\}$. As u_k belongs to $C^1(\mathbb{R}^+;P_k\mathcal{V}_\sigma)$, the supremum T_k is positive for all k. We have, for all T in $[0,T_k]$,

$$\|u_k\|_{L^4([0,T];\mathcal{V}_\sigma)} \leq 2\Big(\sum_{j\in\mathbb{N}}\mu_j(u_0|e_j)^2\Big(\frac{1-e^{-4\nu\mu_j^2 T}}{\nu}\Big)^{\frac12}\Big)^{\frac12}. \tag{5.6}$$

In the case of small data, it is enough to observe that, for any positive T,

$$\sum_{j\in\mathbb{N}}\mu_j(u_0|e_j)^2\Big(\frac{1-e^{-4\nu\mu_j^2T}}{\nu}\Big)^{\frac12} \leq \frac{1}{\nu^{\frac12}}\|u_0\|^2_{\mathcal{V}_\sigma^{\frac12}}.$$

Thus, if $\|u_0\|_{\mathcal{V}_\sigma^{\frac12}} \leq \dfrac{\nu}{8C}$, we have, for any T smaller than T_k, $\|u_k\|_{L^4([0,T];\mathcal{V}_\sigma)} \leq \dfrac{\nu^{\frac34}}{4C}$. This implies that $T_k=+\infty$ and that

$$\|u_k\|_{L^4(\mathbb{R}^+;\mathcal{V}_\sigma)} \leq \frac{2}{\nu^{\frac14}}\|u_0\|_{\mathcal{V}_\sigma^{\frac12}}.$$

In the case of large data, let us define the smallest integer j_0 such that

$$\Big(\sum_{j>j_0}\mu_j(u_0|e_j)^2\Big)^{\frac12} \leq \frac{\nu}{16C}. \tag{5.7}$$

Then, using the fact that $1-e^{-x}\leq x$ for all non negative x, we can write for all $T<T_k$,

$$\begin{aligned}
U_1(T) &\overset{\text{def}}{=} \Big(\sum_{j\in\mathbb{N}}\mu_j(u_0|e_j)^2\Big(\frac{1-e^{-4\nu\mu_j^2T}}{\nu}\Big)^{\frac12}\Big)^{\frac12}\\
&\leq \frac{\nu^{\frac34}}{16C}+\Big(\sum_{j\leq j_0}\mu_j(u_0|e_j)^2\Big(\frac{1-e^{-4\nu\mu_j^2T}}{\nu}\Big)^{\frac12}\Big)^{\frac12}\\
&\leq \frac{\nu^{\frac34}}{16C}+\mu_{j_0}\sqrt{2}\,T^{\frac14}\|u_0\|_{L^2}.
\end{aligned}$$

Thus, stating

$$T_{u_0} \overset{\text{def}}{=} \Big(\frac{\nu^{\frac34}}{16\sqrt{2}C\mu_{j_0}\|u_0\|_{L^2}}\Big)^4,$$

we have, for any positive T less than $\min\{T_k;T_{u_0}\}$,

$$\|u_k\|_{L^4([0,T);\mathcal{V}_\sigma)} \leq \frac{\nu^{\frac34}}{2C}.$$

Thus, for all k, $T_k\geq T_{u_0}$. This implies that $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence of $L^4([0,T];\mathcal{V}_\sigma)$. We infer that a Leray solution u of (NS_ν) exists such that u belongs to $L^4([0,T];\mathcal{V}_\sigma)$. Thanks to Theorem 5.2.1, this solution is unique on $[0,T]$, continuous from $[0,T]$ into $\mathcal{H}$ and satisfies the energy equality on $[0,T]$.

The only thing we have to prove now is the continuity of u from $[0,T]$ into $\mathcal{V}_\sigma^{\frac12}$. As u belongs to $L^4([0,T];\mathcal{V}_\sigma)$, we infer from (5.3) that $Q(u,u)$ is in $L^2([0,T];\mathcal{V}_\sigma^{-\frac12})$. Using Theorem 4.1.1, we get

$$(u(t)|e_j) = (u_0|e_j)e^{-\nu\mu_j^2t}+\int_0^t e^{-\nu\mu_j^2(t-t')}\langle Q(u(t'),u(t')),e_j\rangle\,dt'.$$

Using again (5.3), we infer by definition of the norm on $\mathcal{V}_\sigma^{-\frac{1}{2}}$, that there exists a sequence $(c_j(t))_{j\in\mathbb{N}}$, such that

$$|(u(t)|e_j)| \leq |(u_0|e_j)|e^{-\nu\mu_j^2 t} + C\mu_j^{\frac{1}{2}}\int_0^t e^{-\nu\mu_j^2(t-t')}c_j(t')\|u(t')\|^2_{\mathcal{V}_\sigma}dt'$$

with $\displaystyle\sum_{j\in\mathbb{N}} c_j^2(t) = 1$. Using the Cauchy-Schwarz inequality, we have,

$$\|(u(t)|e_j)\|_{L^\infty([0,T])} \leq |(u_0|e_j)| + \frac{C}{\nu^{\frac{1}{2}}}\mu_j^{-\frac{1}{2}}\left(\int_0^T c_j^2(t)\|u(t)\|^4_{\mathcal{V}_\sigma}\,dt\right)^{\frac{1}{2}}.$$

Multiplying by $\mu_j^{\frac{1}{2}}$ and taking the ℓ^2 norm gives

$$\begin{aligned}
U_2(T) &\stackrel{\text{def}}{=} \left(\sum_{j\in\mathbb{N}}\mu_j\|(u(\cdot)|e_j)\|^2_{L^\infty([0,T])}\right)^{\frac{1}{2}}\\
&\leq \sqrt{2}\|u_0\|_{\mathcal{V}_\sigma^{\frac{1}{2}}} + \sqrt{2}\frac{C}{\nu^{\frac{1}{2}}}\left(\sum_{j\in\mathbb{N}}\int_0^T c_j^2(t)\|u(t)\|^4_{\mathcal{V}_\sigma}\,dt\right)^{\frac{1}{2}}\\
&\leq \sqrt{2}\|u_0\|_{\mathcal{V}_\sigma^{\frac{1}{2}}} + \frac{C}{\nu^{\frac{1}{2}}}\|u\|^2_{L^4([0,T];\mathcal{V}_\sigma)}.
\end{aligned}$$

This gives that u is in $L^\infty([0,T];\mathcal{V}_\sigma^{\frac{1}{2}})$. In fact, it will imply the continuity using the following argument. Let η be any positive number. An integer j_0 exists such that

$$\left(\sum_{j>j_0}\mu_j\|(u(\cdot)|e_j)\|^2_{L^\infty([0,T])}\right)^{\frac{1}{2}} < \frac{\eta}{2}.$$

Now, it turns out that for all $(t_1,t_2)\in[0,T]^2$, one has

$$\begin{aligned}
\|u(t_1)-u(t_2)\|_{\mathcal{V}_\sigma^{\frac{1}{2}}} &\leq \left(\sum_{j>j_0}\mu_j\|(u(\cdot)|e_j)\|^2_{L^\infty([0,T])}\right)^{\frac{1}{2}} + \left(\sum_{j\leq j_0}\mu_j(u(t_1)-u(t_2)|e_j)^2\right)^{\frac{1}{2}}\\
&\leq \frac{\eta}{2} + \mu_{j_0}^{\frac{1}{2}}\|u(t_1)-u(t_2)\|_{L^2}.
\end{aligned}$$

Theorem 4.1.1 tells us that u is continuous from $[0,T]$ into $\mathcal{H}$. Thus the whole Theorem 5.3.1 is proved.

5.3.3 Some remarks about stable solutions

In this paragraph, we shall assume that the bulk force f is identically 0. We shall establish some results about the maximal existence time of the solution constructed in the preceding paragraph.

Proposition 5.3.2 *Let us assume that the initial data u_0 belongs to $\mathcal{V}_\sigma$. Then the maximal time of existence $T^\star$ of the solution u in the space $C([0,T^\star[;\mathcal{V}_\sigma^{\frac{1}{2}})\cap L^4_{loc}([0,T^\star[;\mathcal{V}_\sigma)$ satisfies*

$$T^\star \geq \frac{c\nu^3}{\|\nabla u_0\|^4_{L^2}}.$$

Proof of Proposition 5.3.2. Thanks to (5.6), the maximal time of existence $T^\star$ is bounded from below by T such that

$$4\sum_j \mu_j (u_0|e_j)^2 \Big(\frac{1-e^{-\nu\mu_j^2 T}}{\nu}\Big)^{\frac12} \leq c\nu^{\frac32}.$$

As $1-e^{-\nu\mu_j^2 T} \leq \nu\mu_j^2 T$, we infer that

$$\begin{aligned} 4\sum_j \mu_j (u_0|e_j)^2 \Big(\frac{1-e^{-\nu\mu_j^2 T}}{\nu}\Big)^{\frac12} & \leq 4T^{\frac12}\sum_j \mu_j^2 (u_0|e_j)^2 \\ & \leq 4T^{\frac12}\|u_0\|_{\mathcal{V}_\sigma}^2. \end{aligned}$$

This proves the proposition.

From this proposition, we infer the following corollary.

Corollary 5.3.1 *Let $T^\star$ be the maximal time of existence for a solution u of the system (NS_ν) in the space $C([0,T^\star);\mathcal{V}_\sigma^{\frac12})\cap L^4_{loc}([0,T^\star);\mathcal{V}_\sigma)$. If $T^\star$ is finite, then*

$$\int_0^{T^\star} \|\nabla u(t)\|_{L^2}^4 dt = +\infty \quad \text{and} \quad T^\star \leq \frac{c}{\nu^5}\|u_0\|_{L^2}^4.$$

Proof of Corollary 5.3.1. For almost every t, $u(t)$ belongs to $\mathcal{V}_\sigma$. Then, thanks to the above proposition, the maximal time of existence of the solution starting at time t, which is of course $T^\star - t$, satisfies

$$T^\star - t \geq \frac{c\nu^3}{\|\nabla u(t)\|_{L^2}^4}\cdot$$

This can be written as

$$\|\nabla u(t)\|_{L^2}^4 \geq \frac{c\nu^3}{T^\star - t}\cdot$$

This gives the first part of the corollary. Taking the square root of the above inequality gives, thanks to the energy estimate,

$$c\nu^{\frac52}\int_0^{T^\star} \frac{dt}{(T^\star-t)^{\frac12}} \leq \frac12\|u_0\|_{L^2}^2.$$

The corollary is proved.

Remarks

- The sections 5.1 et 5.3 are to savoir.
- Si vous voulez en savoir plus, vous pouvez consulter les livres of P. Constantin et C. Foias *Navier-Stokes equations*, Chigago University Press, 1988, de P.-G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem.* Chapman & Hall/CRC, Research Notes in Mathematics, **431**, 2002 ou the livre of J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Mathematical Geophysics; an introduction to rotating fluids and Navier-Stokes equations*, Oxford Lecture series in Mathematics and its maps, **32**, Oxford University Press, 2006.

Chapter 6

Linear symmetric systems

6.1 Definition and examples

Let us first define the concept in the framework of linear system with variable coefficients. Let I be a closed interval of $\mathbb{R}$ which has 0 as an interior point. Let us consider a family $\mathcal{A}$ of smooth bounded functions $(\mathcal{A}_k)_{0\leq k\leq d}$ from $I\times\mathbb{R}^d$ into the space of $N\times N$ matrices with real coefficients. Let us assume that all their derivatives in the x variable are bounded. We consider the system

$$(LS)\qquad \begin{cases} \partial_t U + \displaystyle\sum_{k=1}^d \mathcal{A}_k\partial_k U + \mathcal{A}_0 U = F\\ U_{|t=0} = U_0. \end{cases}$$

Let us introduce the following notations. If U in $\mathcal{D}'(\mathbb{R}^d)$ and φ in $\mathcal{D}(\mathbb{R}^d)$,

$$\langle U,\varphi\rangle = \sum_{i=1}^N \langle U^i,\varphi^i\rangle$$

and if U and V belongs to $(L^2(\mathbb{R}^d))^N$,

$$(U|V)_{L^2} = \sum_{i=1}^N \int_{\mathbb{R}^d} U^i(x)V^i(x)dx.$$

Definition 6.1.1 *A function U in $(C(I;L^2))^N$ is a solution of (LS) if and only of for any $\varphi\in\mathcal{D}(\mathbb{R}\times\mathbb{R}^d)$, for any $t\in I$,*

$$\begin{aligned}\langle U(t),\varphi(t)\rangle &= \int_0^t \langle U(t'),\partial_t\varphi(t')\rangle dt' + \langle U_0,\varphi(0)\rangle\\ &+\int_0^t \sum_{k=1}\langle(\mathcal{A}_kU)(t'),\partial_k\varphi(t')\rangle dt' + \int_0^t\langle(\mathrm{div}\,\mathcal{A}U)(t'),\varphi(t')\rangle dt' + \int_0^t\langle F(t'),\varphi(t')\rangle dt'.\end{aligned}$$

Let us define the concept of symmetric system.

Definition 6.1.2 *The above system (LS) is symmetric if and only if for any $k\in\{1,\dots,d\}$ and any $(t,x)\in I\times\mathbb{R}^d$ the matrices $\mathcal{A}_k(t,x)$ are symmetric, which means that for any k, we have $\mathcal{A}_{k,i,j}(t,x)=\mathcal{A}_{k,j,i}(t,x)$.*

An example of such a system is given by (1.2) page 4. The reason why this definition is fundamental is the following. Let us consider a solution of (LS) and let us look to the evolution of its energy. This question leads to the following formal computation which will be made rigourous in the following section.

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|_{L^2}^2 = -\sum_{k=1}^{d}\Big(\mathcal{A}_k\partial_k U\big|U\Big)_{L^2} - (\mathcal{A}_0 U|U)_{L^2} + (F|U)_{L^2}$$

By integration by part, we get that

$$\begin{aligned} -\Big(\mathcal{A}_k\partial_k U\big|U\Big)_{L^2} &= -\sum_{i,j}\int_{\mathbb{R}^d}\mathcal{A}_{k,i,j}\partial_k U^i U^j dx \\ &= \sum_{i,j}\int_{\mathbb{R}^d}\mathcal{A}_{k,i,j}U^i\partial_k U^j dx + \sum_{i,j}\int_{\mathbb{R}^d}\partial_k\mathcal{A}_{k,i,j}U^i U^j dx. \end{aligned}$$

If the system (LS) is symmetric, then we have

$$-\sum_{k=1}^{d}\Big(\mathcal{A}_k\partial_k U\big|U\Big)_{L^2} = \frac{1}{2}((\operatorname{div}\mathcal{A})U|U)_{L^2} \quad \text{with} \quad (\operatorname{div}\mathcal{A})_{i,j} \stackrel{\text{def}}{=} \sum_{k=1}^{d}\partial_k\mathcal{A}_{k,i,j}$$

This implies that

$$\left|\sum_{k=1}^{d}\Big(\mathcal{A}_k(t)\partial_k U(t)\big|U(t)\Big)_{L^2}\right| \leq \frac{1}{2}\|\operatorname{div}\mathcal{A}(t)\|_{L^\infty}\|U(t)\|_{L^2}^2.$$

Thus we get that

$$\frac{d}{dt}\|U(t)\|_{L^2}^2 \leq a_0(t)\|U\|_{L^2}^2 + (F|U)_{L^2} \quad \text{with} \quad a_0(t) \stackrel{\text{def}}{=} \|\operatorname{div}\mathcal{A}(t,\cdot)\|_{L^\infty} + 2\|\mathcal{A}_0(t,\cdot)\|_{L^\infty}. \quad (6.1)$$

The purpose of this section is to study linear symmetric systems. First, we want to solve them and then to study basic properties of their solutions. In this section, for $s\in\mathbb{N}$ we shall state

$$|U(t)|_s^2 \stackrel{\text{def}}{=} \sum_{\substack{1\leq j\leq N\\ 1\leq|\alpha|\leq d}} \|\partial_x^\alpha U^j(t)\|_{L^2}^2.$$

6.2 The wellposedness of linear symmetric systems

The goal of this paragraph is the proof of the following wellposedness theorem.

Theorem 6.2.1 *Let (LS) be a linear symmetric system. Then, if U_0 belongs to H^s and if F is a continuous function with value in H^s, then a unique solution of (S) exists in the space $C^0(I,H^s)\cap C^1(I;H^{s-1})$.*

The proof of this theorem requires four steps

- We first prove a-priori estimates for smooth enough solutions of the system (S).
- Then we apply Friedrichs method.

- Then we pass to the limit in the case of smooth enough initial data and we get existence in any case by smoothing of the initial data.

- Finally, we get uniqueness using existence for the adjoint system.

A priori estimates use in a crucial way the symmetry hypothesis and are true only for smooth enough solutions.

Lemma 6.2.1 *For any non negative integer s, a locally bounded function a_s exists such that for any function U in $C^0(I,H^{s+1})\cap C^1(I,H^s)$, we have for any t in I,*

$$|U(t)|_s \leq |U(0)|_s \exp\int_0^t a_s(t')dt' + \int_0^t |F(t')|_s \exp\left(\int_{t'}^t a_s(t'')dt''\right)dt',$$

with

$$F = \partial_t U + \sum_{k=1}^d \mathcal{A}_k\partial_k U + \mathcal{A}_0 u\,.$$

To start with, let us prove this lemma for $s=0$. Let us consider a function U in the space $C^0(I;H^1)\cap C^1(I;L^2)$. By definition of F, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|U(t)|_0^2 &= (\partial_t U|U)_{L^2}\\ &= (F|U)_{L^2} - (\mathcal{A}_0 U|U)_{L^2} - \sum_{k=1}^d (\mathcal{A}_k\partial_k U|U)_{L^2}\,.\end{aligned}$$

As the system (LS) is symmetric and U belongs to $C^0(I;H^1)\cap C^1(I;L^2)$, computations done page 64 which lead to (6.1) are rigourous. Thus we have

$$\frac{d}{dt}|U(t)|_0^2 \leq a_0(t)|U(t)|_0^2 + 2|F(t)|_0|U(t)|_0 \tag{6.2}$$

with $a_0(t) \overset{\text{def}}{=} \|\operatorname{div}\mathcal{A}(t,\cdot)\|_{L^\infty} + 2\|\mathcal{A}_0(t,\cdot)\|_{L^\infty}$. By Gronwall lemma, we get

$$|U(t)|_0 \leq |U(0)|_0 \exp\int_0^t a_0(t')dt' + \int_0^t |F(t')|_0 \exp\left(\int_{t'}^t a_0(t'')dt''\right)dt'. \tag{6.3}$$

Remark Let us point out that the above computations are also valid when the matrices $(\mathcal{A}_k)$ have C^1 coefficients and $\mathcal{A}_0$ has C^0 coefficients.

Let us study the case when s is any non negative integer. To do so, we shall proceed by induction on the integer s. Let us assume that Lemma 6.2.1 is proved for some s. Let U be a function in $C^0(I,H^{s+2})\cap C^1(I;H^{s+1})$. Let us introduce the function (with $N(d+1)$ components) $\widetilde{U}$ defined by

$$\widetilde{U} = (U,\partial_1 U,\cdots,\partial_d U)\,.$$

As, for any $j\in\{1,\ldots,d\}$,

$$F = \partial_t U + \sum_{k=1}^d \mathcal{A}_k\partial_k U + \mathcal{A}_0 U,$$

we obtain by differentation of the equation,

$$\partial_t\left(\partial_j U\right)=-\sum_{k=1}^d \mathcal{A}_k \partial_k \partial_j U-\sum_{k=1}^d\left(\partial_j \mathcal{A}_k\right) \cdot \partial_k U-\partial_j(\mathcal{A}_0 U)-\partial_j F.$$

We may write

$$\begin{aligned} \partial_t \widetilde{U} &= -\sum_{k=1}^d \mathcal{B}_k\, \partial_k \widetilde{U} - \mathcal{B}_0 \widetilde{U}+\widetilde{F} \quad \text{with} && (6.4)\\ \widetilde{F} &= (F, \partial_1 F, \cdots, \partial_d F) \quad \text{and} && (6.5)\\ \mathcal{B}_k &= \begin{pmatrix} \mathcal{A}_k & & 0 \\ & \mathcal{A}_k & \\ 0 & & \mathcal{A}_k \end{pmatrix}. && (6.6)\end{aligned}$$

The coefficients of $\mathcal{B}_0$ are computed from those of $\mathcal{A}_k$ $(k=0,\ldots,d)$ and their first order derivatives. The induction hypothesis allows to conclude the proof of Lemma 6.2.1.

Remark Let us point out that the proof of the inequalities of Lemma 6.2.1 done above demands exactly one more derivative than in the statement of Theorem 6.2.1.

This leads us to use a smoothing method, the Friedrich method. It consists in smoothing both the initial data and the system itself. More precisely, let us consider the system (LS_n) defined by

$$(LS_n) \quad \begin{cases} \partial_t U_n + \displaystyle\sum_{k=1}^d \mathbb{E}_n\left(\mathcal{A}_k \partial_k U_n\right) + E_n(\mathcal{A}_0 U_n) = E_n F \\ \mathbb{E}_n\, U_{|t=0} = \mathbb{E}_n\, U_0 \end{cases}$$

where $\mathbb{E}_n$ is the cutoff operator defined on L^2 by

$$\mathbb{E}_n\, u \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{B(0,n)} \widehat{u}). \tag{6.7}$$

This is nothing else that the orthogonal projection of L^2 on the closed space L^2_n of the L^2 functions the Fourier transform of which are supported in the ball of center 0 and radius n. Lemma 7.1.1 tells us in particular that the operator ∂_k is a continuous on L^2_n. As the functions $\mathcal{A}_k$ are bounded it turns out that the linear operator

$$V \longmapsto \sum_{k=1}^d \mathbb{E}_n\left(\mathcal{A}_k \partial_k V\right)+\mathbb{E}_n(\mathcal{A}_0 V)$$

is continuous on L^2_n. Thus the system (LS_n) is a linear sytstem of ordinary differential equations on L^2_n. This implies the existence of a unique function U_n continuous on I with value in L^2_n which is a solution of (LS_n). Moreover as the functions $\mathcal{A}_k$ are smooth functions in (t,x), using the equation (LS_n), we get that U_n is a smooth function on I with value in H^s for any integer s.

Let us prove that the functions U_n satisfy the energy estimates of Lemma 6.2.1. More precisely, we have the following lemma.

Lemma 6.2.2 *For any non negative integer s, a locally bounded function a_s exists such that for any $n \in N$ and any t in I we have,*

$$|U_n(t)|_s \leq |\,\mathbb{E}_n\, U(0)|_s \exp \int_0^t a_s(t')dt' + \int_0^t |\,\mathbb{E}_n\, F(t')|_s \exp\left(\int_{t'}^t a_s(t'')dt''\right) dt',$$

Taking the scalar product of (LS_n) with U_n in L^2, we get using the fact that the operator $\mathbb{E}_n$ is selfadjoint on L^2 and that $\mathbb{E}_n U_n = U_n$,

$$\frac{d}{dt}|U_n(t)|_0^2 = -2\sum_{k=1}^d (\mathcal{A}_k \partial_k U_n | U_n)_{L^2} - 2(\mathcal{A}_0 U_n)|U_n)_{L^2} - 2(\mathbb{E}_n F | U_n)_{L^2}.$$

We proceed exactly as in the proof of Lemma 6.2.1. As the system (LS) is symmetric and U_n belongs to $C^0(I;H^1)\cap C^1(I;L^2)$, computations done page 64 which lead to (6.1) are rigourous. Thus we have

$$\frac{d}{dt}\,|U_n(t)|_0^2 \leq a_0(t)|U_n(t)|_0^2 + 2|\,\mathbb{E}_n F(t)|_0|U_n(t)|_0 \tag{6.8}$$

with $a_0(t) \stackrel{\text{def}}{=} \|\operatorname{div}\mathcal{A}(t,\cdot)\|_{L^\infty} + 2\|\mathcal{A}_0(t,\cdot)\|_{L^\infty}$. Gronwall Lemma implies that

$$|U_n(t)|_0 \leq |\,\mathbb{E}_n U_0|_0 \exp\int_0^t a_0(t')dt' + \int_0^t |\,\mathbb{E}_n F(t')|_0 \exp\left(\int_{t'}^t a_0(t'')dt''\right) dt'.$$

The proof of the lemma for any integer s works exactly as the one of Lemma 6.2.1 and is omitted.

The third step consists in the proof of the following wellposedness result.

Proposition 6.2.1 *Let $s \geq 3$. We consider the linear symmetric system*

$$(LS)\quad \begin{cases} \partial_t U + \displaystyle\sum_{k=1}^d \mathcal{A}_k\,\partial_k U \;+ \mathcal{A}_0 U = F \\ U(0) = U_0 \end{cases}$$

with F in $C(I;H^s)$ and U_0 in H^s. A unique solution U exists in $C(I,H^{s-2})\cap C^1(I;H^{s-3})$ which moreover satisfies the energy estimate

$$\forall \sigma \leq s\,,\ \forall\, t\in I\,,\ |U(t)|_\sigma \leq |U_0|_\sigma\, \exp\int_0^t a_s(t')dt' + \int_0^t |F(t')|_\sigma \exp\left(\int_{t'}^t a_s(t'')dt''\right) dt'.$$

Let us consider the sequence $(U_n)_{n\in\mathbb{N}}$ of solution of (LS_n). We shall prove that $(U_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(I;H^{s-2})$. In order to do so, let us state $V_{n,p} \stackrel{\text{def}}{=} U_{n+p} - U_n$. We have

$$\begin{cases} \partial_t V_{n,p} + \displaystyle\sum_{k=1}^d \mathbb{E}_{n+p}\left(\mathcal{A}_k\,\partial_k V_{n,p}\right) \;+ \mathbb{E}_{n+p}(\mathcal{A}_0 V_{n,p}) = F_{n,p} \\ V_{n,p}(0) = (\mathbb{E}_{n+p} - \mathbb{E}_n)U_0 \end{cases} \tag{6.9}$$

with

$$F_{n,p} \stackrel{\text{def}}{=} \sum_{k=1}^d (\mathbb{E}_{n+p} - \mathbb{E}_n)\left(\mathcal{A}_k\,\partial_k U_n\right) \,-(\mathbb{E}_{n+p} - \mathbb{E}_n)(\mathcal{A}_0 U_n) + (\mathbb{E}_{n+p} - \mathbb{E}_n)F.$$

Lemma 6.2.2 tells us that the sequence $(U_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(I;H^s)$. Moreover we have

$$|(\mathbb{E}_{n+p} - \mathbb{E}_n)a|_{\sigma-1} \leq \frac{C}{n}|a|_\sigma.$$

Thus we have

$$\begin{aligned}
|(\mathbb{E}_{n+p}-\mathbb{E}_n)\,(\mathcal{A}_k\,\partial_k U_n(t))|_{s-2} &\leq \frac{C}{n}\sup_k|(\mathbb{E}_{n+p}-\mathbb{E}_n)\,(\mathcal{A}_k\,\partial_k U_n(t))|_{s-1}\\
&\leq \frac{C}{n}\,|U_n(t)|_s.
\end{aligned}$$

The sames arguments give

$$\Big|(\mathbb{E}_{n+p}-\mathbb{E}_n)(\mathcal{A}_0 U_n(t))+(\mathbb{E}_{n+p}-\mathbb{E}_n)F(t)\Big|_{s-2}\leq\frac{C}{n^2}\,(|U_n(t)|_s+|F(t)|_s). \tag{6.10}$$

Energy estimate implies that

$$|V_{n,p}(t)|_{s-2}\leq\frac{C}{n}\,\exp\Big(t\int_0^t a_s(t')dt'\Big).$$

Thus the sequence $(U_n)_{n\in\mathbb{N}}$ is a Cauchy one in $L^\infty(I,H^{s-2})$. Moreover, using (6.9) and (6.10), we infer that $(\partial_t U_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(I;H^{s-3})$. Let us denote by U the limit of $(U_n)_{n\in\mathbb{N}}$. Of course, U belongs to $C(I;H^{s-2})\cap C^1(I;H^{s-3})$. Let us check that this function U is solution of (LS). As F belongs to $C(I;H^s)$, we have that

$$\lim_{n\to\infty}\mathbb{E}_n\,F=F\quad\text{in}\quad L^\infty(I;H^s). \tag{6.11}$$

As the sequence $(U_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(I;H^s)$, we have that

$$\|(\mathbb{E}_n-\mathrm{Id})\mathcal{A}_k(U)\partial_k U_n\|_{L^\infty(I;H^{s-2})}\leq\frac{C}{n}\cdot$$

Thus U is a solution of (LS). To conclude, let us point out that the sequence $(U_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(I;H^s)$. Using interpolation inequality, we get that for any $s'<s$, the sequence $(U_n)_{n\in\mathbb{N}}$ is a Cauchy one in $C(I,H^{s'})$. Thus U belongs to $C(I,H^{s'})$. Using the fact that U is a solution of (LS), we get that U belongs to $C(I,H^{s'})\cap C^1(I;H^{s'-1})$. But as $(U_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(I;H^s)$, it weakly converges to U in $L^\infty(I;H^s)$. Using (6.11), the fact that $(\mathbb{E}_n\,U_0)_{n\in\mathbb{N}}$ converges to U_0 in H^s and that

$$\|U\|_{L^\infty([0,t];H^s)}\leq\limsup_{n\to\infty}\|U_n\|_{L^\infty([0,t];H^s)}$$

we get, passing to the limit in Lemma 6.2.2 that

$$|U_n(t)|_s\leq|\,\mathbb{E}_n\,U(0)|_s\exp\int_0^t a_s(t')dt'+\int_0^t|\,\mathbb{E}_n\,F(t')|_s\exp\Big(\int_{t'}^t a_s(t'')dt''\Big)\,dt'.$$

Proposition 6.2.1 is proved.

Let us conclude the existence part of Theorem 6.2.1. In order to do so, let us consider the sequence $(\widetilde{U}_n)_{n\in\mathbb{N}}$ of solutions of

$$\left\{\begin{array}{l}
\dfrac{\partial\widetilde{U}_n}{\partial t}+\displaystyle\sum_{k=1}^d\mathcal{A}_k\partial_k\widetilde{U}_n+\mathcal{A}_0\widetilde{U}_n=\mathbb{E}_n\,F\\
\widetilde{U}_{n|t=0}=\mathbb{E}_n\,U_0.
\end{array}\right.$$

Thanks to Proposition 6.2.1 this solution does exist in $C^1(I, H^s)$ for any positive real number s. Let us state $V_{n,p} \stackrel{\text{def}}{=} U_{n+p} - U_n$. It satisfies

$$\begin{cases} \partial_t \widetilde{V}_{n,p} + \displaystyle\sum_{k=1}^{d} \mathcal{A}_k \partial_k \widetilde{V}_{n,p} + \mathcal{A}_0 \widetilde{V}_{n,p} = (\mathbb{E}_{n+p} - \mathbb{E}_n)F \\ \widetilde{V}_{n,p|t=0} = (\mathbb{E}_{n+p} - \mathbb{E}_n)U_0. \end{cases}$$

Lemma 6.2.1 implies that

$$|\widetilde{V}_{n,p}(t)|_s \leq |(\mathbb{E}_{n+p} - \mathbb{E}_n)U(0)|_s \exp \int_0^t a_s(t')dt' \\ + \int_0^t |(\mathbb{E}_{n+p} - \mathbb{E}_n)F(t')|_s \exp\left(\int_{t'}^t a_s(t')dt'\right) dt.$$

As the function F is continuous from I into H^s, the sequence $(\mathbb{E}_n F)_{n\in\mathbb{N}}$ converges to F in the space $L^\infty([0,T]; H^s)$. As U_0 belongs to H^s, the sequence $(\mathbb{E}_n U_0)_{n\in\mathbb{N}}$ converges to U_0 in H^s. Thus the sequence $(\widetilde{U}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(I; H^s)$. It converges to a function U of $C(I; H^s)$ which is of course solution of the system (LS). The fact that $\partial_t U$ belongs to $C(I; H^{s-1})$ comes immediately from the fact that U is solution of (S).

The existence part of Theorem 6.2.1 and also the uniqueness when $s \geq 1$ is now proved. The following proposition will conclude the proof of Theorem 6.2.1.

Proposition 6.2.2 *Let U be a solution $C^0(I; L^2)$ of the symmetric system (LS).*

$$(LS) \quad \begin{cases} \partial_t U + \displaystyle\sum_{k=1}^{d} \mathcal{A}_k \, \partial_k U \, + \mathcal{A}_0 U = 0 \\ U_{|t=0} = 0. \end{cases}$$

Then $U \equiv 0$.

In order to prove this proposition, we shall use a duality method. Let ψ be a function of $\mathcal{D}(]0,T[\times \mathbb{R}^d)$. Let us consider the solution of

$$({}^tLS) \quad \begin{cases} -\partial_t \varphi - \displaystyle\sum_{k=1}^{d} \partial_k(\mathcal{A}_k \varphi) + {}^t\mathcal{A}_0 \varphi = \psi \\ \varphi_{|t=T} = 0. \end{cases}$$

The system $({}^tLS)$ can be understood as the adjoint system of the system (LS). As we have

$$\partial_k(\mathcal{A}_k \varphi) = \mathcal{A}_k \partial_k \varphi + \partial_k \mathcal{A}_k \varphi,$$

the system $({}^tLS)$ becomes

$$({}^tS) \quad \begin{cases} -\partial_t \varphi - \displaystyle\sum_{k=1}^{d} \mathcal{A}_k \partial_k \varphi + \widetilde{\mathcal{A}}_0 \varphi = \psi \\ \varphi_{|t=T} = 0 \end{cases} \qquad \text{with} \quad \widetilde{\mathcal{A}}_0 \stackrel{\text{def}}{=} {}^t\mathcal{A}_0 - \sum_{k=1}^{d} \partial_k \mathcal{A}_k.$$

This is obviously a linear symmetric system. The existence part of Theorem 6.2.1 tells us that a solution φ of $({}^tLS)$ exists in $C^1(I,H^s)$ for any s in $\mathbb{N}$. Thus we have

$$\begin{aligned}\langle U,\psi\rangle &= \Big\langle U, -\partial_t\varphi - \sum_{k=1}^d \mathcal{A}_k\partial_k\varphi + \widetilde{\mathcal{A}}_0\varphi\Big\rangle\\ &= -\int_I \langle U(t,\cdot),\partial_t\varphi(t,\cdot\rangle dt - \sum_{k=1}^d\int_I\langle U(t),\partial_k(\mathcal{A}_k\varphi)(t)\rangle dt\\ &\qquad\qquad + \int_{I\times\mathbb{R}^d} U(t,x)^t\mathcal{A}_0\varphi(t,x)dtdx.\end{aligned}$$

Considering the weak regularity of U, each integration by part must be justified. Using Theorem 6.3.2 page 72 below (the proof of which is totally independant of Proposition 6.2.2), we have that for any t in I, the function $\varphi(t,\cdot)$ belongs to $\mathcal{D}(\mathbb{R}^d)$. By definition of the derivative of distributions, we have

$$\begin{aligned}\langle U(t),\partial_k(\mathcal{A}_k\varphi)(t)\rangle &= \sum_{i,j}\langle U^i(t),\partial_k(\mathcal{A}_{k,i,j}\varphi^j)(t)\rangle\\ &= \sum_{i,j}\langle\partial_k U^i(t),\mathcal{A}_{k,i,j}\varphi^j(t)\rangle.\end{aligned}$$

Because the matrices $\mathcal{A}_k$ are symmetric, we have for any t in I,

$$\langle U(t),\partial_k(\mathcal{A}_k\varphi)(t)\rangle = \langle\mathcal{A}_k\frac{\partial U(t)}{\partial x_k},\varphi(t)\rangle.$$

It turns ot that

$$\langle U,\psi\rangle = -\int_I\langle U(t,\cdot),\partial_t\varphi(t,\cdot\rangle dt - \Big\langle\sum_{k=1}^d\mathcal{A}_k\partial_k U - \widetilde{\mathcal{A}}_0U,\varphi\Big\rangle.$$

In order to justify the time integration by part, let us observe that U belongs to $C^1(I,H^{-1})$. Indeed, as for smooth function we have

$$\begin{aligned}\langle\mathcal{A}_k\partial_k V,\varphi\rangle &= -\langle V,\partial_k{}^t\mathcal{A}_k\varphi\rangle - \langle V,{}^t\mathcal{A}_k\partial_k\varphi\rangle\\ &\leq (\|\mathcal{A}_k(t,\cdot)\|_{L^\infty}+a_0(t))\|V\|_{L^2}\|\varphi\|_{H^1}.\end{aligned}$$

This implies that $\partial_t U$ belongs to $C^0(I;H^{-1})$. Now, let us use the smoothing operator $\mathbb{E}_n$ defined by (6.7). The function $\mathbb{E}_n U$ belongs to $C^1(I;H^s)$ for any s. Using this with s greater than $d/2$ implies that for any x, the function

$$t\mapsto\mathbb{E}_n U(t,x)$$

exists and is a C^1 function on I. This impies that

$$\begin{aligned}-\int_I\mathbb{E}_n U(t,x)\frac{\partial\varphi}{\partial t}(t,x)dt &= -\mathbb{E}_n U(T,x)\varphi(T,x) + \mathbb{E}_n U(0,x)\varphi(0,x)\\ &\qquad + \int_I\frac{\partial\,\mathbb{E}_n U}{\partial t}(t,x)\varphi(t,x)dt.\end{aligned}$$

Using the fact that $U_0=0$ and that $\varphi(T,\cdot)=0$, we get that

$$-\int_I\mathbb{E}_n U(t,x)\partial_t\varphi(t,x)dt = \int_I\partial_t(\mathbb{E}_n U)(t,x)\varphi(t,x)dt.$$

By integration in the variable x and interchanging time and space integration, we get that

$$-\int_I \langle \mathbb{E}_n U(t,\cdot), \partial_t\varphi(t,\cdot)\rangle dt = \int_I \langle \partial_t(\mathbb{E}_n U)(t,\cdot), \varphi(t,\cdot)\rangle dt.$$

As U is a function of $C(I;L^2)\cap C^1(I;H^{-1})$, we have

$$\lim_{n\to\infty} \mathbb{E}_n U = U \quad \text{in} \quad L^\infty(I,L^2) \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}_n \partial_t U = \partial_t U \quad \text{in} \quad L^\infty(I,H^{-1}).$$

Passing to the limit in the above equality gives

$$-\langle U, \partial_t\varphi\rangle = \int_I \langle \partial_t U(t,\cdot), \varphi(t,\cdot)\rangle dt$$

and thus

$$\langle U, \psi\rangle = \int_I \Big\langle \partial_t U(t,\cdot) + \sum_{k=1}^d \mathcal{A}_k\partial_k U(t,\cdot) + \mathcal{A}_0 U(t,\cdot), \varphi(t,\cdot)\Big\rangle.$$

But as U is solution of (LS) with $F=0$, then $U\equiv 0$ which ends the proof of the proposition and thus the one of Theorem 6.2.1.

6.3 Finite propagation speed

The phenomena of finite propagation speed is describe by the following theorem.

Theorem 6.3.1 *Let (LS) be a symmetric system. A constant C_0 exists such that, for any positive real number R and any data $F\in C^0(I,L^2)$ and $U_0\in L^2$ such that*

$$F(t,x)\equiv 0 \quad \textit{when} \quad |x|<R-C_0t \quad \textit{and} \quad U_0(x)\equiv 0 \quad \textit{when} \quad |x|<R. \tag{6.12}$$

then the unique solution U of the system (LS) in $C^0(I,L^2)$ with data F and U_0 satisfies

$$U(t,x)\equiv 0 \quad \textit{when} \quad |x|<R-C_0t.$$

An other form of this statement is given by the following corollary.

Corollary 6.3.1 *If the data F and U_0 satisfy*

$$F(t,x)\equiv 0 \quad \textit{for} \quad |x|>R+C_0t \quad \textit{and} \quad U_0(x)\equiv 0 \quad \textit{for} \quad |x|>R,$$

then the solution U satisfies

$$U(t,x)\equiv 0 \quad \textit{when} \quad |x|>R+C_0t.$$

To start with, let us regularize the data U_0 and F perturbing their support as less as possible. Let χ be a function of $\mathcal{D}(B(0,1))$ the integral of which is 1. For any positive ϵ, we state

$$\chi_\varepsilon(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon^d}\chi\left(\frac{x}{\varepsilon}\right)\cdot$$

Now let us consider the data

$$U_{0,\varepsilon} \stackrel{\text{def}}{=} \chi_\varepsilon \star U_0 \quad \text{and} \quad F_\varepsilon(t,\cdot) \stackrel{\text{def}}{=} \chi_\varepsilon \star F(t,\cdot).$$

Of course, we have

$$\text{Supp } U_{0,\varepsilon} \subset \text{Supp } U_0 + B(0,\varepsilon) \quad \text{and} \quad F_\varepsilon(t,\cdot) \subset \text{Supp } F(t,\cdot) + B(0,\varepsilon).$$

The support hypothesis are satisfies for $U_{0,\varepsilon}$ and F_ε with $R+\varepsilon$ instead of R and the associated solution U_ε is $C^1(I;H^s)$ for any $s \in \mathbb{N}$. Thus it is enough to prove Theorem 6.3.1 with those regular solutions, namely the following statement.

Theorem 6.3.2 *Let (LS) be a symmetric system. A constant C_0 exists such that, for any positive real number R and any data $F \in C^0(I,H^1) \cap C^1(I,L^2)$ and $U_0 \in H^1$ such that*

$$F(t,x) \equiv 0 \quad \text{when} \quad |x| < R - C_0 t \quad \text{and} \quad U_0(x) \equiv 0 \quad \text{when} \quad |x| < R. \tag{6.13}$$

then the unique solution U of the system (LS) in $C^0(I,H^1) \cap C^1(I,L^2)$ with data F and U_0 satisfies

$$U(t,x) \equiv 0 \quad \text{when} \quad |x| < R - C_0 t.$$

In order to do so, we use weighted energy estimate. More precisely, for τ greater than 1, let us introduce

$$U_\tau(t,x) \stackrel{\text{def}}{=} e^{\tau\phi(t,x)} U(t,x).$$

with $\phi(t,x) = -t + \psi(x)$. The function ψ is a smooth real valued function on $\mathbb{R}^d$ which will be choosen later on.

$$\partial_t U_\tau + \sum_{k=1}^d \mathcal{A}_k \, \partial_k U_\tau + \mathcal{B}_\tau U_\tau = F_\tau$$

with

$$\begin{aligned} F_\tau(t,x) &\stackrel{\text{def}}{=} e^{\tau\phi(t,x)} F(t,x) \quad \text{and} \\ \mathcal{B}_\tau &\stackrel{\text{def}}{=} \mathcal{A}_0 + \tau \left(\partial_t \phi \text{Id} + \sum_{k=1}^d \partial_k \phi \mathcal{A}_k \right) \end{aligned}$$

Considering the form of the function ϕ, we have

$$\mathcal{B}_\tau = \mathcal{A}_0 - \tau \left(\text{Id} - \sum_{k=1}^d \partial_k \psi \mathcal{A}_k \right)$$

Thus a constant $K > 0$ exists such that for any $(t,x) \in I \times \mathbb{R}^d$, any vector $W \in \mathbb{R}^N$ and any positive real number τ, we have

$$\|\nabla\psi\|_{L^\infty} \leq K \Rightarrow (\mathcal{B}_\tau(t,x) W | \bar{W}) \leq (\mathcal{A}_0(t,x) W | W).$$

Then let us write the energy estimate and use the above inequality and relation (6.1); we get

$$\begin{aligned} \frac{d}{dt} |U_\tau(t)|_0^2 &= -2 \sum_{k=1}^d (\mathcal{A}_k \partial_k U | U_\tau)_{L^2} - 2(\mathcal{B}_\tau U_\tau | U_\tau)_{L^2} + 2(F_\tau | U_\tau)_{L^2} \\ &\leq a_0(t) |U_\tau|_0^2 + (F_\tau | U_\tau)_{L^2} \end{aligned}$$

Using Gronwall Lemma, we get

$$|U_\tau(t)|_0 \leq |U_\tau(0)|_0 \exp\int_0^t a_0(t')dt' + \int_0^t |F_\tau(t')|_0 \exp\Big(\int_{t'}^t a_0(t'')dt''\Big)dt'. \tag{6.14}$$

Let us point out that the above inequality is independant of τ. Now let us state $C_0 = 1/K$ and let us pick up a smooth function $\psi = \psi(|x|)$ such that

$$-2\varepsilon + K(R-|x|) \leq \psi(x) \leq -\varepsilon + K(R-|x|) \quad \text{and} \quad \|\nabla\psi\|_{L^\infty} \leq K. \tag{6.15}$$

Then we have

$$\forall (t,x) \in I\times\mathbb{R}^d\,,\ |x| \geq R - C_0 t \Longrightarrow -t+\psi(x) \leq -\varepsilon.$$

When τ tends to $+\infty$ in the inequality (6.14), we get that

$$\forall t\in I\,,\ \lim_{\tau\rightarrow\infty}\int_{\mathbb{R}^d} e^{2\tau\phi(t,x)}|u(t,x)|^2dx = 0.$$

Thus $U(t,x)\equiv 0$ on the open set $t<\psi(x)$. But, if (t_0,x_0) satisfies $|x_0| < R - C_0t_0$, it is possible to pick up a function ψ satisfying (6.15) and such that $t_0 < \psi(x_0)$. This proves the theorem.

6.4 A final remark about Gronwall's lemma

In this chapter, we often use Gronwall lemma. Let us state and prove a genreral version of it.

Lemma 6.4.1 *Let us consider f a C^1 function and a and b two functions from an open interval I of $\mathbb{R}$ (which contains 0) into $\mathbb{R}^+$. Let us assume that*

$$(f^2)' \leq af^2 + bf.$$

Then, we have, for any positive t,

$$f(t) \leq f(0)\exp\Big(\frac{1}{2}\int_0^t a(t')dt'\Big) + \int_0^t b(t')\exp\Big(\frac{1}{2}\int_{t'}^t a(t'')'dt'\Big)dt'.$$

Proof Let us define

$$f_a(t) \stackrel{\text{def}}{=} f(t)\exp\Big(\frac{1}{2}\int_0^t a(t')dt'\Big) \quad \text{and} \quad b_a(t) \stackrel{\text{def}}{=} b(t)\exp\Big(\frac{1}{2}\int_0^t a(t')dt'\Big).$$

We have

$$(f_a^2(t))' = \big((f^2(t))' - a(t)f^2(t)\big)\exp\Big(\frac{1}{2}\int_0^t a(t')dt'\Big) \leq b_a(t)f_a(t).$$

By integration, we get

$$f_a^2(t) \leq f^2(0) + \int_0^t b_a(t')f_a(t')dt'.$$

Defining $M_a(t) \stackrel{\text{def}}{=} \sup\limits_{0\leq t'\leq t} f_a(t')$, we infer that

$$M_a^2(t) \leq f^2(0) + M_a(t)\int_0^t b_a(t')dt'.$$

Thus we have

$$\Big(M_a(t) - \frac{1}{2}\int_0^t b_a(t')dt'\Big)^2 \leq f^2(0) + \frac{1}{4}\Big(\int_0^t b_a(t')dt'\Big)^2 \leq \Big(f(0) + \frac{1}{2}\int_0^t b_a(t')dt'\Big)^2.$$

We deduce that

$$M_a(t) \leq f(0) + \int_0^t b_a(t')dt',$$

which concludes the proof of the lemma.

6.5 Réferences and remarques

Ce chapitre is to connaître to l'exception of the section 6.3. .

Chapter 7

Littlewood-Paley theory

7.1 Localization in frequency space

The very basic idea of this theory consists in a localization procedure in the frequency space. The interest of this method is that the derivatives (or more generally the Fourier multipliers) act in a very special way on distributions the Fourier transform of which is supported in a ball or a ring. More precisely, we have the following lemma.

7.1.1 Bernstein inequalities

Lemma 7.1.1 (of localization) *Let $\mathcal{C}$ be a ring, B a ball. A constant C exists so that, for any non negative integer k, any smooth homogeneous function σ of degree m, any couple of real (a,b) so that $b \geq a \geq 1$ and any function u of L^a, we have*

$$\text{Supp } \widehat{u} \subset \lambda B \Rightarrow \sup_{\alpha = k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1} \lambda^{k + d(\frac{1}{a} - \frac{1}{b})} \|u\|_{L^a};$$
$$\text{Supp } \widehat{u} \subset \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^a} \leq \sup_{\alpha = k} \|\partial^\alpha u\|_{L^a} \leq C^{k+1} \lambda^k \|u\|_{L^a};$$

Proof of Lemma 7.1.1 Using a dilation of ifze λ, we can assume all along the proof that $\lambda = 1$. Let ϕ be a function of $\mathcal{D}(\mathbb{R}^d)$ the value of which is 1 near B. As $\widehat{u}(\xi) = \phi(\xi)\widehat{u}(\xi)$, we can write, if g denotes the inverse fourier transform of ϕ,

$$\partial^\alpha u = \partial^\alpha g \star u.$$

Applying Young inequalities the result follows through

$$\begin{aligned} \|\partial^\alpha g\|_{L^c} &\leq \|\partial^\alpha g\|_{L^\infty} + \|\partial^\alpha g\|_{L^1} \\ &\leq 2\|(1 + |\cdot|^2)^d \partial^\alpha g\|_{L^\infty} \\ &\leq 2\|(\mathrm{Id} - \Delta)^d \big((\cdot)^\alpha \phi\big)\|_{L^1} \\ &\leq C^{k+1}. \end{aligned}$$

To prove the second assertion, let us consider a function $\widetilde{\phi}$ which belongs to $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$ the value of which is identically 1 near the ring $\mathcal{C}$. Using the algebraic Using the following algebraic identity

$$\begin{aligned} |\xi|^{2k} &= \sum_{1 \leq j_1, \cdots, j_k \leq d} \xi_{j_1}^2 \cdots \xi_{j_k}^2 \\ &= \sum_{|\alpha| = k} (i\xi)^\alpha (-i\xi)^\alpha, \end{aligned} \tag{7.1}$$

and stating $g_\alpha \stackrel{\text{def}}{=} \mathcal{F}^{-1}(i\xi_j)^\alpha|\xi|^{-2k}\widetilde{\phi}(\xi)$, we can write, as $\widehat{u} = \widetilde{\phi}\widehat{u}$ that

$$\widehat{u} = \sum_{|\alpha|=k} (-i\xi)^\alpha \widehat{g}_\alpha \widehat{u},$$

which implies that

$$u = \sum_{|\alpha|=k} g_\alpha \star \partial^\alpha u \tag{7.2}$$

and then the result. This proves the whole lemma.

7.1.2 Dyadic partition of unity

Now, let us define a dyadic partition of unity. We shall use it all along this text.

Proposition 7.1.1 *Let us define by $\mathcal{C}$ the ring of center 0, of small radius 3/4 and great radius 8/3. It exists two radial functions χ and φ the values of which are in the interval $[0,1]$, belonging respectively to $\mathcal{D}(B(0,4/3))$ and to $\mathcal{D}(\mathcal{C})$ such that*

$$\forall \xi \in \mathbb{R}^d\,,\ \chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \tag{7.3}$$

$$\forall \xi \in \mathbb{R}^d \setminus\{0\}\,,\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \tag{7.4}$$

$$|j-j'| \geq 2 \Rightarrow \text{Supp }\ \varphi(2^{-j}\cdot) \cap \text{Supp }\ \varphi(2^{-j'}\cdot) = \emptyset, \tag{7.5}$$

$$q \geq 1 \Rightarrow \text{Supp }\ \chi \cap \text{Supp }\ \varphi(2^{-j}\cdot) = \emptyset, \tag{7.6}$$

If $\widetilde{\mathcal{C}} = B(0,2/3) + \mathcal{C}$, then $\widetilde{\mathcal{C}}$ is a ring and we have

$$|j-j'| \geq 5 \Rightarrow 2^{j'}\widetilde{\mathcal{C}} \cap 2^j\mathcal{C} = \emptyset, \tag{7.7}$$

$$\forall \xi \in \mathbb{R}^d\,,\ \frac{1}{3} \leq \chi^2(\xi) + \sum_{j\geq 0} \varphi^2(2^{-j}\xi) \leq 1, \tag{7.8}$$

$$\forall \xi \in \mathbb{R}^d \setminus\{0\}\,,\ \frac{1}{2} \leq \sum_{j\in\mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1. \tag{7.9}$$

Proof of Proposition 7.1.1 Let us choose α in the interval $]1,4/3[$ let us denote by $\mathcal{C}'$ the ring of small radius α^{-1} and big radius 2α. Let us choose a smooth function θ, radial with value in $[0,1]$, supported in $\mathcal{C}$ with value 1 in the neighbourhood of $\mathcal{C}'$. The important point is the following. For any couple of integers (p,q) we have

$$|j-j'| \geq 2 \Rightarrow 2^j\mathcal{C} \cap 2^{j'}\mathcal{C} = \emptyset. \tag{7.10}$$

Let us suppose that $2^{j'}\mathcal{C} \cap 2^j\mathcal{C} \neq \emptyset$ and that $p \geq q$. It turns out that $2^{j'} \times 3/4 \leq 4 \times 2^{j+1}/3$, which implies that $j'-j \leq 1$. Now let us state

$$S(\xi) = \sum_{j\in\mathbb{Z}} \theta(2^{-j}\xi).$$

Thanks to (7.10), this sum is locally finite on the space $\mathbb{R}^d \backslash \{0\}$. Thus the function S is smooth on this space. As α is greater than 1,

$$\bigcup_{j\in\mathbf{Z}} 2^j \mathcal{C}' = \mathbb{R}^d \backslash \{0\}.$$

As the function θ is non negative and has value 1 near $\mathcal{C}'$, it comes from the above covering property that the above function is positive. Then let us state

$$\varphi = \frac{\theta}{S}. \tag{7.11}$$

Let us check that φ fits. It is obvious that $\varphi \in \mathcal{D}(\mathcal{C})$. The function $1 - \sum_{j\geq 0} \varphi(2^{-j}\xi)$ is smooth thanks to (7.10). As the support of θ is included in $\mathcal{C}$, we have

$$|\xi| \geq \frac{4}{3} \Rightarrow \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1. \tag{7.12}$$

thus stating

$$\chi(\xi) = 1 - \sum_{j\geq 0} \varphi(2^{-j}\xi), \tag{7.13}$$

we get Identites (7.3)and (7.5). Identity (7.6) is a obvious consequence of (7.10) and of (7.12). Now let us prove (7.7) which will be useful in Section 7.4. It is clear that the ring $\widetilde{\mathcal{C}}$ is the ring of center 0, of small radius 1/12 and of big radius 10/3. Then it turns out that

$$2^{j'}\widetilde{\mathcal{C}} \cap 2^j\mathcal{C} \neq \emptyset \Rightarrow \Big(\frac{3}{4} \times 2^j \leq 2^{j'} \times \frac{10}{3} \quad \text{ou} \quad \frac{1}{12} \times 2^{j'} \leq 2^j \frac{8}{3}\Big),$$

and (7.7) is proved. Now let us prove (7.8). As χ and φ have their values in $[0,1]$, it is clear that

$$\chi^2(\xi) + \sum_{j\geq 0} \varphi^2(2^{-j}\xi) \leq 1. \tag{7.14}$$

Let us bound from below the sum of squares. The notation $a \equiv b(2)$ means that $a - b$ is even. So we have

$$1 = (\chi(\xi) + \Sigma_0(\xi) + \Sigma_1(\xi))^2 \quad \text{with}$$
$$\Sigma_0(\xi) = \sum_{j\equiv 0(2), q\geq 0} \varphi(2^{-j}\xi) \quad \text{and} \quad \Sigma_1(\xi) = \sum_{j\equiv 1(2), q\geq 0} \varphi(2^{-j}\xi).$$

From this it comes that $1 \leq 3(\chi^2(\xi) + \Sigma_0^2(\xi) + \Sigma_1^2(\xi))$. But thanks to (7.5), we get

$$\Sigma_i^2(\xi) = \sum_{j\geq 0, q\equiv i(2)} \varphi^2(2^{-j}\xi)$$

and the proposition is proved.

We shall consider all along this book two fixed functions χ and φ satisfying the assertions (7.3)–(7.8). Now let us to fix the notations that will be used in all the following of this text.

Notations

$$h=\mathcal{F}^{-1}\varphi \quad \text{and} \quad \widetilde{h}=\mathcal{F}^{-1}\chi,$$
$$\Delta_{-1}u=\chi(D)u=\mathcal{F}^{-1}(\chi(\xi)\widehat{u}(\xi)),$$
$$\text{if} \quad j\geq 0\,,\ \Delta_j u=\varphi(2^{-j}D)u=2^{jd}\int_{\mathbb{R}^d}h(2^jy)u(x-y)dy,$$
$$\text{if} \quad j\leq -2\,,\ \Delta_j u=0,$$
$$S_j u=\sum_{j'\leq j-1}\Delta_{j'}u=\chi(2^{-j}D)u=2^{jd}\int_{\mathbb{R}^d}\widetilde{h}(2^jy)u(x-y)dy,$$
$$\text{if} \quad j\in\mathbb{Z}\,,\ \dot{\Delta}_j u=\varphi(2^{-j}D)u=2^{jd}\int_{\mathbb{R}^d}h(2^jy)u(x-y)dy,$$
$$\text{if} \quad j\in\mathbb{Z}\,,\ \dot{S}_j u=\sum_{j'\leq j-1}\dot{\Delta}_{j'}u.$$

Remark Let us point that all the above operators Δ_j and S_j maps L^p into L^p with norms which do not depend on j. This fact will be used all along this book.

Now let us have a look of the case when we may write

$$\text{Id}=\sum_j \Delta_j \quad \text{or} \quad \text{Id}=\sum_j \dot{\Delta}_j.$$

This is described by the following proposition.

Proposition 7.1.2 *Let u be in $\mathcal{S}'(\mathbb{R}^d)$. Then, we have, in the sense of the convergence in the space $\mathcal{S}'(\mathbb{R}^d)$,*

$$u=\lim_{j\to\infty}S_j u.$$

Proof of Proposition 7.1.2 Let $f\in\mathcal{S}(\mathbb{R}^d)$. We have $< u-S_ju, f > = < u, f-S_jf >$. Thus it is enough to prove that in the space $\mathcal{S}(\mathbb{R}^d)$, we have

$$f=\lim_{j\to\infty}S_j f.$$

We shall use the family of semi norms $\|\cdot\widehat{\|}_{k,\mathcal{S}}$ of $\mathcal{S}$ defined by

$$\|f\widehat{\|}_{k,\mathcal{S}}\stackrel{\text{def}}{=}\sup_{\substack{|\alpha|\leq k\\ \xi\in\mathbb{R}^d}}(1+|\xi|)^k|\partial^\alpha\widehat{f}(\xi)|.$$

Thanks to Leibnitz formula, we have

$$\|f-S_jf\widehat{\|}_{k,\mathcal{S}}\leq\sup_{\substack{|\alpha|\leq k\\ \xi\in\mathbb{R}^d}}\Big\{(1+|\xi|)^k\Big(|1-\chi(2^{-j}\xi)|\times|\partial^\alpha\widehat{f}(\xi)|)$$
$$+\sum_{0<\beta\leq\alpha}C_\alpha^\beta 2^{-q|\beta|}|(\partial^\beta\chi)(2^{-j}\xi)|\times|\partial^{\alpha-\beta}\widehat{f}(\xi)|\Big)\Big\}.$$

As χ equals to 1 near the origin it turns out that

$$\|f-S_jf\widehat{\|}_{k,\mathcal{S}}\leq C_\alpha 2^{-j}\|f\widehat{\|}_{k+1,\mathcal{S}}.$$

The proposition is proved.

The following proposition tells us that the condition of convergence in $\mathcal{S}'$ is somehow weak for series, the Fourier transform of which is supported in dyadic rings.

Proposition 7.1.3 *Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of bounded functions such that the Fourier transform of u_j is supported in $2^j\widetilde{\mathcal{C}}$ where $\widetilde{\mathcal{C}}$ is a given ring. Let us assume that*

$$\|u_j\|_{L^\infty} \leq C2^{jN}.$$

Then the series $(u_j)_{j\in\mathbb{N}}$ is convergent in $\mathcal{S}'$.

Proof of Proposition 7.1.3 Let us use the relation (7.2). After rescaling it can be written as

$$u_j = 2^{-jk} \sum_{|\alpha|=k} 2^{jd} g_\alpha(2^j\cdot) \star \partial^\alpha u_j.$$

Then for any test function ϕ in $\mathcal{S}$, let us write that

$$\begin{aligned} \langle u_j, \phi\rangle &= -2^{-jk} \sum_{|\alpha|=k} \langle u_j, 2^{jd} g_\alpha(2^j\cdot) \star \partial^\alpha \phi\rangle && (7.15)\\ &\leq C2^{-jk} \sum_{|\alpha|=k} 2^{jN} \|\partial^\alpha \phi\|_{L^1}. \end{aligned}$$

Let us choose $k > N$. Then $(\langle u_j, \phi\rangle)_{j\in\mathbb{N}}$ is a convergent series, the sum of which is less than $C\|\phi\|_{M,\mathcal{S}}$ for some integer M. Thus the formula

$$\langle u, \phi\rangle \stackrel{\text{def}}{=} \lim_{j\to\infty} \sum_{j'\leq j} \langle \Delta_{j'} u, \phi\rangle$$

defines a tempered distribution.

7.2 Inhomogeneous Besov spaces

7.2.1 Definition and examples

Definition 7.2.1 *Let s be a real number, and p and r two reals numbers greater than 1. The Besov spaces $B^s_{p,r}$ is the space of all tempered distributions so that*

$$\|u\|_{B^s_{p,r}} \stackrel{\text{def}}{=} \left\|(2^{js}\|\Delta_j u\|_{L^p})_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} < +\infty.$$

The following proposition (the proof of which is straighforward and omitted) describes the relations between homogeneous and inhomogeneous spaces.

Proposition 7.2.1 *Let s be a negative number. Then $\dot{B}^s_{p,r}$ is a subset of $B^s_{p,r}$ and a constant C (independent of s) exists so that, for any u belonging to $\dot{B}^s_{p,r}$, we have*

$$\|u\|_{B^s_{p,r}} \leq \frac{C}{-s}\|u\|_{\dot{B}^s_{p,r}}.$$

Let s be a positive number. Then $B^s_{p,r}$ is a subset of $\dot{B}^s_{p,r}$ when p is finite, $B^s_{\infty,r}\cap\mathcal{S}'_h$ is a subset of $\dot{B}^s_{\infty,r}$ and a constant C exists (independent of s) so that, for any u belonging to $B^s_{p,r}$, we have

$$\|u\|_{\dot{B}^s_{p,r}} \leq \frac{C}{s}\|u\|_{B^s_{p,r}}.$$

Lemma 7.2.1 *If r is finite, then for any u in $B^s_{p,r}$, we have*

$$\lim_{j\to\infty} \|S_j u - u\|_{B^s_{p,r}} = 0$$

The proof of this proposition is an easy exercise left to the reader. Let us give the first example for Besov space, the Sobolev spaces H^s. We have the following result.

Theorem 7.2.1 *The two spaces H^s and $B^s_{2,2}$ are equal and the two norms satisfies*

$$\frac{1}{C^{|s|+1}}\|u\|_{B^s_{2,2}} \leq \|u\|_{H^s} \leq C^{|s|+1}\|u\|_{B^s_{2,2}}.$$

As the support of the Fourier transform of $\Delta_j u$ is included in the ring $2^j\mathcal{C}$, it is clear, as $j \geq 0$, that a constant C exists such that, for any real s and any u such that $\widehat{u}$ belongs to L^2_{loc},

$$\frac{1}{C^{|s|+1}}2^{js}\|\Delta_j u\|_{L^2} \leq \|\Delta_j u\|_{H^s} \leq C^{|s|+1}2^{js}\|\Delta_j u\|_{L^2}. \tag{7.16}$$

Using Identity (7.8), we get

$$\frac{1}{3}\|u\|^2_{H^s} \leq \int \chi^2(\xi)(1+|\xi|^2)^s|\widehat{u}(\xi)|^2 d\xi + \sum_{j\geq 0}\int \varphi^2(2^{-j}\xi)(1+|\xi|^2)^s|\widehat{u}(\xi)|^2 d\xi \leq \|u\|^2_{H^s}$$

which proves the theorem.

Proposition 7.2.2 *The space $B^0_{p,1}$ is continuously embedded in L^p and the space L^p is continuously embedded in $B^0_{p,\infty}$.*

The proof is trivial. The first includedion comes from the fact that the series $(\Delta_j u)_{j\in\mathbb{Z}}$ is convergent in L^p. The second one comes from the fact that for any p, we have $\|\Delta_j u\|_{L^p} \leq C\|u\|_{L^p}$.

7.2.2 Basic properties

The first point to look at is the invariance with respect to the choice of the dyadic partition of unity choosen to define the space. Most of the properties of the Besov spaces are based on the following lemma.

Lemma 7.2.2 *Let $\mathcal{C}'$ be a ring in $\mathbb{R}^d$; let s be a real number and p and r two real numbers greater than 1. Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of smooth functions such that*

$$\text{Supp } \widehat{u}_j \subset 2^j\mathcal{C}' \quad \textit{and} \quad \left\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{N}}\right\|_{\ell^r} < +\infty.$$

Then we have

$$u = \sum_{j\in\mathbb{N}} u_j \in B^s_{p,r} \quad \textit{and} \quad \|u\|_{B^s_{p,r}} \leq C_s\left\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{N}}\right\|_{\ell^r}.$$

This immediately implies the following corollary.

Corollary 7.2.1 *The space $B^s_{p,r}$ does not depend on the choice of the functions χ and φ used in the Definition 7.2.1.*

In order to prove the lemma, let us first observe that $(u_j)_{j\in\mathbb{N}}$ is a convergente series in $\mathcal{S}'$. Indeed using Lemma 7.1.1, we get that $\|u_j\|_{L^\infty} \leq C2^{j\left(\frac{d}{p}-s\right)}$. Proposition 7.1.3 implies that $(u_j)_{j\in\mathbb{N}}$ is a convergente series in $\mathcal{S}'$. Then, let us study $\Delta_{j'}u$. As $\mathcal{C}$ and $\mathcal{C}'$ are two rings, an integer N_0 exists so that

$$|j'-j| \geq N_0 \Longrightarrow 2^j\mathcal{C} \cap 2^{j'}\mathcal{C}' = \phi.$$

Here $\mathcal{C}$ is the ring defined in the Proposition 7.1.1. Now, it is clear that

$$\begin{aligned} |j'-j| \geq N_0 & \Longrightarrow \mathcal{F}(\Delta_{j'}u_j) = 0 \\ & \Longrightarrow \Delta_{j'}u_j = 0. \end{aligned}$$

Now, we can write that

$$\begin{aligned} \|\Delta_{j'}u\|_{L^p} & = \|\sum_{|j-j'|<N_0} \Delta_{j'}u_j\|_{L^p} \\ & \leq C \sum_{|j-j'|<N_0} \|u_j\|_{L^p}. \end{aligned}$$

So, we obtain that

$$\begin{aligned} 2^{j's}\|\Delta_{j'}u\|_{L^p} & \leq C \sum_{\substack{j'\geq -1 \\ |j-j'|\leq N_0}} 2^{j's}\|u_j\|_{L^p} \\ & \leq C \sum_{\substack{j'\geq -1 \\ |j-j'|\leq N_0}} 2^{js}\|u_j\|_{L^p}. \end{aligned}$$

We deduce from this that

$$2^{j's}\|\Delta_{j'}u\|_{L^p} \leq (c_k)_{k\in\mathbb{Z}} \star (d_\ell)_{\ell\in\mathbb{Z}} \quad \text{with} \quad c_k = \mathbf{1}_{[-N_0,N_0]}(k) \quad \text{and} \quad d_\ell = \mathbf{1}_{\mathbb{N}}(\ell)2^{\ell s}\|u_\ell\|_{L^p}.$$

The classical property of convolution between $\ell^1(\mathbb{Z})$ and $\ell^r(\mathbb{Z})$ gives that

$$\|u\|_{B^s_{p,r}} \leq C\biggl(\sum_{j\in\mathbb{N}} 2^{rqs}\|u_j\|^r_{L^p}\biggr)^{\frac{1}{r}},$$

which proves the lemma.

The following theorem is the equivalent of Sobolev embedding (see Theorem **??** page **??**).

Theorem 7.2.2 *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then for any real number s the space $B^s_{p_1,r_1}$ is continuously embedded in $B^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}$.*

In order to prove this result, we again apply Lemma 7.1.1 which tells us that

$$\begin{aligned} \|S_0u\|_{L^{p_2}} & \leq C\|u\|_{L^{p_1}} \quad \text{and} \\ \|\Delta_j u\|_{L^{p_2}} & \leq C2^{jd\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|\Delta_j u\|_{L^{p_1}}. \end{aligned}$$

Considering that $\ell^{r_1}(\mathbb{Z}) \subset \ell^{r_2}(\mathbb{Z})$, the theorem is proved.

Proposition 7.2.3 *The space $B^s_{p,r}$ is continuously embedded in $\mathcal{S}'$.*

By definition $B^s_{p,r}$ is a subspace of $\mathcal{S}'$. Thus we have only to proof of a constant C and an integer M exists such that for any test function ϕ in $\mathcal{S}$ we have

$$\langle u, \phi\rangle \leq C\|u\|_{B^s_{p,r}}\|\phi\|_{M,\mathcal{S}}. \tag{7.17}$$

Using the above Theorem 7.2.2 and the relation (7.15), we can write, if N is a large enough integer,

$$\begin{aligned}
\langle \Delta_j u, \phi\rangle &= -2^{-j(N+1)} \sum_{|\alpha|=N+1} \langle \Delta_j u, 2^{jd} g_\alpha(2^j\cdot) \star \partial^\alpha \phi\rangle \\
&\leq 2^{-j}\|u\|_{B^{-N}_{\infty,\infty}} \sup_{|\alpha|=N+1} \|\partial^\alpha\phi\|_{L^1} \\
&\leq C2^{-j}\|u\|_{B^s_{p,r}}\|\phi\|_{M,\mathcal{S}}.
\end{aligned} \tag{7.18}$$

Now Proposition 7.1.2 implies the Inequality (7.17).

Theorem 7.2.3 *The space $B^s_{p,r}$ equipped with the norm $\|\cdot\|_{B^s_{p,r}}$ is a Banach space and satisfies the Fatou properties, i.e. if $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of $B^s_{p,r}$, then an element u of $B^s_{p,r}$ and a subsequence $u_{\psi(n)}$ exist such that*

$$\lim_{n\to\infty} u_n = u \text{ in } \mathcal{S}' \quad \text{and} \quad \|u\|_{B^s_{p,r}} \leq \liminf_{n\to\infty} \|u_n\|_{B^s_{p,r}}.$$

Proof of Theorem 7.2.3 Let us first prove the Fatou property. Using Lemma 7.1.1, we claim that, for any j, the sequence $(\Delta_j u_n)_{n\in\mathbb{N}}$ is bounded in $L^p \cap L^\infty$. Then, using Cantor's diagonal process, we infer the existence of a subsequence $(u_{\psi(n)})_{n\in\mathbb{N}}$ and a sequence $(\widetilde{u}_j)_{j\in\mathbb{Z}}$ such that, for any $j \in \mathbb{Z}$ and any $\phi \in \mathcal{S}$,

$$\lim_{n\to\infty} \int \Delta_j u_{\psi(n)}(x)\phi(x)dx = \int \widetilde{u}_j(x)\phi(x)dx \quad \text{and} \quad \|\widetilde{u}_j\|_{L^p} \leq \lim_{n\to\infty} \|\Delta_j u_n\|_{L^p}.$$

As the Fourier transform of $(\Delta_j u_n)_{n\in\mathbb{N}}$ is supported in $2^j\widetilde{\mathcal{C}}$, the same holds for $\widetilde{u}_j$. Then, let us observe that the sequence $\big((2^{js}\|\Delta_j u_n\|_{L^p})_j\big)_{n\in\mathbb{N}}$ is a bounded sequence of ℓ^r, an element $(\widetilde{c}_j)_j$ of ℓ^r such that (up to an omitted extraction), we have, for any sequence $(d_j)_j$ of non negative real numbers different from 0 only for a finite number of indices j,

$$\lim_{n\to\infty} \sum_j 2^{js}\|\Delta_j u_{\psi(n)}\|_{L^p} d_j = \sum_j \widetilde{c}_j d_j \quad \text{and} \quad \|(\widetilde{c}_j)_j\|_{\ell^r} \leq \lim_{n\to\infty} \|u_{\psi(n)}\|_{\dot{B}^s_{p,r}}.$$

Passing to the limit in the sum gives that $(2^{js}\|\widetilde{u}_j\|_{L^p})_j$ belongs to $\ell^r(\mathbb{Z})$. Using Lemma 7.1.1 and Proposition 7.2.2 implies that the series $(\widetilde{u}_j)_{j\in\mathbb{Z}}$ converges to some u in $B^s_{p,r}$ such that

$$\|u\|_{B^s_{p,r}} \leq C_s\Big\|(2^{js}\|\widetilde{u}_j\|_{L^p})_j\Big\|_{\ell^r}.$$

This is proves the first part of the theorem.

Now, let us check that $\dot{B}^s_{p,r}$ is complete. Let us consider a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$. This sequence is of course bounded. Thus u exists in $\dot{B}^s_{p,r}$ such that a subsequence $(u_{\psi(n)})_{n\in\mathbb{N}}$ converges to u in $\mathcal{S}'$. Using that, for any positive ε, an integer n_ϵ exists such that

$$\forall (n,m) \in [n_\varepsilon, \infty[\,,\ \|u_m - u_{\psi(n)}\|_{B^s_{p,}} < \varepsilon.$$

Applying the above method to the sequence $(u_m - u_{\psi(n)})_{n\in\mathbb{N}}$, we infer that

$$\forall m \geq n_\varepsilon\,,\ \|u_m - u\|_{\dot{B}^s_{p,r}} \leq \varepsilon.$$

The theorem is proved.

7.3 The case of Hölder type spaces

Another relevant example of Besov spaces are Hölder spaces.

Definition 7.3.1 *Let (k,ρ) be in $\mathbb{N}\times]0,1]$. The* Hölder space $C^{k,\rho}(\mathbb{R}^d)$, *(or $C^{k,\rho}$ if no confusion is possible) is the space of C^k functions u on $\mathbb{R}^d$ such that*

$$\|u\|_{C^{k,\rho}} = \sup_{|\alpha|\leq k}\Big(\|\partial^\alpha u\|_{L^\infty} + \sup_{x\neq y}\frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\rho}\Big) < \infty.$$

We have the following result.

Proposition 7.3.1 *For any k in $\mathbb{N}$, a constant C_k exists such that for any $\rho\in]0,1]$ and any function u belonging to $C^{k,\rho}$, we have*

$$\sup_j 2^{j(k+\rho)}\|\Delta_j u\|_{L^\infty} \leq C_k\|u\|_{C^{k,\rho}}.$$

To prove this, let us first observe that, when $j=-1$, we have that $\|S_0 u\|_{L^\infty}\leq C\|u\|_{L^\infty}$. When j is non negative, let us write the operator Δ_j of the convolution form

$$\Delta_j u(x) = 2^{jd}\int h(2^j(x-y))u(y)dy.$$

The fact that the function φ is identically 0 near the origin implies that for any $\alpha\in\mathbb{N}^d$,

$$\int x^\alpha h(x)dx = 0.$$

Thus we can write

$$\Delta_j u(x) = 2^{jd}\int h(2^j(x-y))\Big(u(y)-u(x) - \sum_{0<|\alpha|\leq k}\frac{1}{\alpha!}\partial^\alpha u(x)(y-x)^\alpha\Big)dy. \tag{7.19}$$

Taylor formula at order $k-1$ implies that

$$\begin{aligned} u(y) - u(x) - &\sum_{0<|\alpha|\leq k}\frac{1}{\alpha!}\partial^\alpha u(x)(y-x)^\alpha \\ &= \int_0^1 k(1-t)^{k-1}\sum_{|\alpha|=k}\frac{(y-x)^\alpha}{\alpha!}\Big(\partial^\alpha u(x+t(y-x)) - \partial^\alpha u(x)\Big)dt. \end{aligned}$$

The fact that the functions $\partial^\alpha u$ belong to the space C^ρ implies that

$$\Big|u(y)-u(x) - \sum_{0<|\alpha|\leq k}\frac{1}{\alpha!}\partial^\alpha u(x)(y-x)^\alpha\Big| \leq C_k|y-x|^{k+\rho}\|u\|_{C^{k,\rho}}.$$

Then it comes from (7.19) that

$$|\Delta_j u(x)| \leq C_k\|u\|_{C^{k,\rho}}2^{jd}\int |x-y|^{k+\rho}|h(2^j(x-y))|dy$$

and the proposition is proved.

Let us study the reciproque of Proposition 7.3.1. We have the following proposition.

Proposition 7.3.2 *Let r be in $\mathbb{R}^+ \setminus \mathbb{N}$ and let u be in $B^r_{\infty,\infty}$. Then u is in $C^{k,\rho}$ with $k = [r]$ and $\rho = r - [r]$. Moreover we have*

$$\|u\|_{C^{k,\rho}} \leq C_r \|u\|_{B^r_{\infty,\infty}} \quad \textit{with} \quad C_r = C_k\biggl(\frac{1}{\rho} + \frac{1}{1-\rho}\biggr).$$

To star with, let us observe that, thanks to Lemma 7.1.1, we have, for any α the length of which is less than r,

$$\|\Delta_j \partial^\alpha u\|_{L^\infty} \leq C^{k+1} 2^{-q(r-|\alpha|)} \|u\|_{B^r_{\infty,\infty}}.$$

Thus the series $(\Delta_j \partial^\alpha u)_{j\in\mathbf{N}}$ is convergent in the space L^∞ and we have

$$\|\partial^\alpha u\|_{L^\infty} \leq C^{k+1} \frac{1}{\rho} \|u\|_{B^r_{\infty,\infty}}. \tag{7.20}$$

Now let us study the derivative of order k. We can write, for a positive integer N which will be chosen later on, that

$$|\partial^\alpha u(x) - \partial^\alpha u(y)| \leq \sum_{j=0}^{N-1} |\partial^\alpha \Delta_j u(x) - \partial^\alpha \Delta_j u(y)| + \sum_{j\geq N} |\partial^\alpha \Delta_j u(x) - \partial^\alpha \Delta_j u(y)|.$$

By Taylor inequality, we have that

$$|\partial^\alpha \Delta_j u(x) - \partial^\alpha \Delta_j u(y)| \leq C|x-y| \sup_{|\beta|=k+1} \|\partial^\beta \Delta_j u\|_{L^\infty}.$$

Using Lemma 7.1.1, we get

$$|\partial^\alpha \Delta_j u(x) - \partial^\alpha \Delta_j u(y)| \leq C_k \|u\|_{B^r_{\infty,\infty}} |x-y| 2^{-q(\rho-1)}. \tag{7.21}$$

The high frequency terms are estimated very roughly writing

$$|\partial^\alpha \Delta_j u(x) - \partial^\alpha \Delta_j u(y)| \leq C_k 2^{-q\rho} \|u\|_{B^r_{\infty,\infty}}.$$

Then it comes from (7.21) that

$$|\partial^\alpha u(x) - \partial^\alpha u(y)| \leq C_k \|u\|_{B^r_{\infty,\infty}} \Bigl(\sum_{j=0}^{N} 2^{-q(\rho-1)} |x-y| + \sum_{j\geq N+1} 2^{-q\rho}\Bigr).$$

Thanks to (7.20), we may assume that $|x-y| \leq 1$. Choosing

$$N = [-\log_2 |x-y|] + 1,$$

in the above inequality, we conclude the proof of the proposition.

Propositions 7.3.1 and 7.3.2 together imply the following theorem.

Theorem 7.3.1 *Let r be in $\mathbb{R}^+ \setminus \mathbb{N}$. Then the spaces $B^r_{\infty,\infty}$ and $C^{[r],r-[r]}$ are equal and we have*

$$C_{[r]}^{-1} \|u\|_{B^r_{\infty,\infty}} \leq \|u\|_{C^{[r],r-[r]}} \leq C_{[r]} \biggl(\frac{1}{r-[r]} + \frac{1}{1-(r-[r])}\biggr) \|u\|_{B^r_{\infty,\infty}}.$$

7.4 Paradifferential computationus

In this section, we are going to study the way how the product acts on Besov spaces. Of course, we shall use the dyadic decomposition constructed in the Section 7.1.2.

7.4.1 Bony's decomposition

Let us consider two tempered distributions u and v, we write

$$u = \sum_{j'} \Delta_{j'} u \quad \text{and} \quad v = \sum_{j} \Delta_j v.$$

Formally, the product can be written as

$$uv = \sum_{j,j'} \Delta_{j'} u \Delta_j v.$$

Now, let us introduce Bony's decomposition.

Definition 7.4.1 *We shall designate paraproduct by u and shall denote by $T_u v$ the following bilinear operator:*

$$T_u v \stackrel{\text{def}}{=} \sum_j S_{j-1} u \Delta_j v.$$

We shall designate remainder of u and v and shall denote by $R(u,v)$ the following bilinear operator:

$$R(u,v) = \sum_{|j-j'|\leq 1} \Delta_{j'} u \Delta_j v.$$

Just by looking at the definition, it is clear that

$$uv = T_u v + T_v u + R(u,v). \tag{7.22}$$

The way how paraproduct and remainder act on Besov spaces is described by the following theorem.

Lemma 7.4.1 *For any s, a constant C exists such that, for any (p,r) in $[1,+\infty]^2$, we have*

$$\forall (u,v) \in L^\infty \times B^s_{p,r}\,,\ \|T_u v\|_{B^s_{p,r}} \leq C\|u\|_{L^\infty} \|v\|_{B^s_{p,r}}.$$

Proof of Lemma 7.4.1 From the assertion (7.7), the Fourier transform of $S_{j-1}u\Delta_j v$ is supported in $2^j\widetilde{\mathcal{C}}$. Then, let us write that

$$\|S_{j-1}u\Delta_j v\|_{L^p} \leq C\|u\|_{L^\infty}\|\Delta_j v\|_{L^p}.$$

Lemma 7.2.2 implies the result.

Now we shall study the behaviour of operators R. Here we have to consider terms of the type $\Delta_j u \Delta_j v$. The Fourier transform of such terms is not supported in ring but in balls of the type $2^j B$. Thus to prove that remainder terms belong to some Besov spaces, we need the following lemma.

Lemma 7.4.2 *Let B be a ball of $\mathbb{R}^d$, s a positive real number and (p,r) in $[1,\infty]^2$. Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of smooth functions such that*

$$\text{Supp } \widehat{u}_j \subset 2^j B \quad \text{and} \quad \left\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{N}}\right\|_{\ell^r} < +\infty.$$

Then we have

$$u = \sum_{j\in\mathbb{N}} u_j \in B^s_{p,r} \quad \text{and} \quad \|u\|_{B^s_{p,r}} \leq C_s \left\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{N}}\right\|_{\ell^r}.$$

Proof of Lemma 7.4.2 We have for any j

$$\|u_j\|_{L^p} \leq C 2^{-qs}.$$

As s is positive, $(u_j)_{j\in\mathbb{N}}$ is a convergent series in L^p. We then study $\Delta_{j'} u_j$. As $\mathcal{C}$ is a ring (defined in the proposition 7.1.1) and B is a ball, an integer N_1 exists so that

$$j' \geq j + N_1 \Longrightarrow 2^{j'}\mathcal{C} \cap 2^j B = \phi.$$

So it is clear that

$$\begin{aligned} j' \geq j + N_1 \quad &\Longrightarrow \quad \mathcal{F}(\Delta_{j'} u_j) = 0 \\ &\Longrightarrow \quad \Delta_{j'} u_j = 0. \end{aligned}$$

Now, we write that

$$\begin{aligned} \|\Delta_{j'} u\|_{L^p} &= \left\| \sum_{j \geq j'-N_1} \Delta_{j'} u_j \right\|_{L^p} \\ &\leq \sum_{j \geq j'-N_1} \|\Delta_{j'} u_j\|_{L^p} \\ &\leq \sum_{j \geq j'-N_1} \|u_j\|_{L^p}. \end{aligned}$$

So, we get that

$$\begin{aligned} 2^{j's}\|\Delta_{j'} u\|_{L^p} &\leq \sum_{j \geq j'-N_1} 2^{j's}\|u_j\|_{L^p} \\ &\leq \sum_{j \geq j'-N_1} 2^{(j'-j)s} 2^{js}\|u_j\|_{L^p}. \end{aligned}$$

So, we deduce from this that

$$2^{j's}\|\Delta_{j'} u\|_{L^p} \leq (c_k) \star (d_\ell) \quad \text{with} \quad c_k = \mathbf{1}_{[-N_1,+\infty[}(k) 2^{-ks} \quad \text{and} \quad d_\ell = 2^{\ell s}\|u_\ell\|_{L^p}.$$

So, the lemma is proved.

Lemma 7.4.3 *For any (s_1, s_2) such that $s_1 + s_2 > 0$ a constant C exists such that , any (p_1, p_2, r_1, r_2) in $[1,\infty]^4$ such that*

$$, \ \frac{1}{p} \stackrel{\text{def}}{=} \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad \text{and} \quad \frac{1}{r_1} + \frac{1}{r_2} \stackrel{\text{def}}{=} \frac{1}{r} \leq 1,$$

we have that

$$\forall (u,v) \in B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}, \ \|R(u,v)\|_{B^{s_1+s_2}_{p,r})} \leq C\|u\|_{B^{s_1}_{p_1,r_1}} \|v\|_{B^{s_2}_{p_2,r_2}}.$$

Proof of Lemma 7.4.3 By definition of the remainder operator, we have

$$R(u,v)=\sum_j R_j \quad\text{with}\quad R_j=\sum_{\ell=-1}^{1}\Delta_{j-\ell}u\Delta_j v.$$

By definition of Δ_j, the support of the Fourier transform of R_j is included in $2^jB(0,24)$. Moreover, Hölder inequalities implies that

$$2^{j(s_1+s_2)}\|R_j\|_{L^p}\leq\sum_{\ell=-1}^{1}\|\Delta_{j-\ell}u\|_{L^{p_1}}\|\Delta_j v\|_{L^{p_2}}.$$

Thus $2^{j(s_1+s_2)}\|R_j\|_{L^p}$ appears to be a sum of three series which are the product of a ℓ^{r_1} series by a ℓ^{r_2} series. Thus the lemma is proved.

Now, we are going to infer the following corollary.

Corollary 7.4.1 *For any positive s, the space $L^\infty\cap B^s_{p,r}$ is an algebra. More precilesely, a constant C exists such that*

$$\|uv\|_{B^s_{p,r}}\leq C(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}}+\|u\|_{B^s_{p,r}}\|v\|_{L^\infty}).$$

The proof is nothing but the use of Bony's decomposition and the map of Theorems 7.4.1 and 7.4.3.

7.4.2 Action of smooth functions

In this paragraph we shall study the action of smooth functions on the space $B^s_{p,r}$. More precisely, if f is a smooth function vanishing at 0, and u a function of $B^s_{p,r}$, does $f\circ u$ belongs to $B^s_{p,r}$? The answer is given by the following theorem.

Theorem 7.4.1 *Let f be a smooth function and s a positive real number and (p,r) in $[1,\infty]^2$. If u belongs to $B^s_{p,r}\cap L^\infty$, then $f\circ u$ belongs to $B^s_{p,r}$ and we have*

$$\|f\circ u\|_{B^s_{p,r}}\leq C(s,f,\|u\|_{L^\infty})\|u\|_{B^s_{p,r}}.$$

Before proving this theorem, let us notice that if $s>d/p$ or if $s=d/p$ and $r=1$, then the space $B^s_{p,r}$ is included into L^∞. Thus in those cases, the space $B^s_{p,r}$ is stable under the action of f by composition. This is in particular the case for the Sobolev space H^s with $s>d/2$.

Let us prove the theorem. We shall use the argument of the so called "telescopic series". As the sequence $(S_ju)_{j\in\mathbb{N}}$ converges to u in L^p and $f(0)=0$, then we have

$$f(u)=\sum_j f_j \quad\text{with}\quad f_j\stackrel{\text{def}}{=}f(S_{j+1}u)-f(S_ju). \tag{7.23}$$

Using Taylor formula at order 1, we get

$$f_j=m_j\Delta_ju \quad\text{with}\quad m_j\stackrel{\text{def}}{=}\int_0^1 f'(S_ju+t\Delta_ju)dt. \tag{7.24}$$

At this point of the proof, let us point out that there is no hope for the Fourier transform of f_j to be compactly supported. Thus Lemma 7.4.2 is not efficient in this case. We shall prove the following improvement of this lemma.

Lemma 7.4.4 *Let s be a positive real number and (p,r) in $[1,\infty]^2$. A constant C_s exists such that if $(u_j)_{j\in\mathbb{N}}$ is a sequence of smooth functions which satisfies*

$$\Big(\sup_{|\alpha|\leq[s]+1} 2^{j(s-|\alpha|)}\|\partial^\alpha u_j\|_{L^p}\Big)_j \in \ell^r,$$

then we have

$$u=\sum_{j\in\mathbb{N}} u_j \in B^s_{p,r} \quad \text{and} \quad \|u\|_{B^s_{p,r}} \leq C_s\Big\|\Big(\sup_{|\alpha|\leq[s]+1} 2^{j(s-|\alpha|)}\|\partial^\alpha u_j\|_{L^p}\Big)_j\Big\|_{\ell^r}.$$

As s is positive, the series $(u_q)_{q\in\mathbb{N}}$ is convergent in L^p. Let us denote its sum by u and let us write that

$$\Delta_j u = \sum_{j'\leq j}\Delta_j u_{j'} + \sum_{j'>q}\Delta_j u_{j'}.$$

Using that $\|\Delta_j u_{j'}\|_{L^p}\leq \|u_{j'}\|_{L^p}$, we get that

$$\begin{aligned} 2^{js}\Big\|\sum_{j'>q}\Delta_j u_{j'}\Big\|_{L^p} &\leq 2^{js}\sum_{j'>q}\|u_{j'}\|_{L^p} \\ &\leq \sum_{j'>q} 2^{-(q'-q)s}2^{j's}\|u_{j'}\|_{L^p}. \end{aligned} \tag{7.25}$$

Then using Lemma 7.1.1, we write that

$$\|\Delta_j u_{j'}\|_{L^p} \leq C2^{-q([s]+1)}\sup_{|\alpha|=[s]+1}\|\partial^\alpha u_{j'}\|_{L^p}.$$

Then we write

$$2^{js}\Big\|\sum_{j'\leq j}\Delta_j u_{j'}\Big\|_{L^p} \leq \sum_{j'\leq j} 2^{(j'-j)([s]+1-s)}\sup_{|\alpha|=[s]+1} 2^{j'(s-|\alpha|)}\|\partial^\alpha u_{j'}\|_{L^p}.$$

This inequality together with (7.25) implies that

$$\begin{aligned} 2^{js}\|\Delta_j u\|_{L^p} &\leq (a\star b)_j \quad \text{with} \\ a_j &\stackrel{\text{def}}{=} \mathbf{1}_{\mathbb{N}}(j)2^{-qs} + \mathbf{1}_{\mathbb{N}}(j)2^{-j([s]+1-s)} \quad \text{and} \\ b_j &\stackrel{\text{def}}{=} 2^{js}\|u_j\|_{L^p} + \sup_{|\alpha|=[s]+1} 2^{j(s-|\alpha|)}\|\partial^\alpha u_j\|_{L^p}. \end{aligned}$$

This proves the lemma.

Let us go back to the proof of Theorem 7.4.1. Let us admit for a while that

$$\forall\alpha\in\mathbb{N}^d,\ \|\partial^\alpha m_j\|_{L^\infty} \leq C_\alpha(f,\|u\|_{L^\infty})2^{j|\alpha|}. \tag{7.26}$$

Thus using Leibnitz formula and Lemma 7.1.1, we get that

$$\|\partial^\alpha f_j\|_{L^p} \leq \sum_{\beta\leq\alpha} C^\alpha_\beta 2^{j|\beta|}C_\beta(f,\|u\|_{L^\infty})2^{j(|\alpha|-|\beta|)}\|\Delta_j u\|_{L^p}$$

Thus we get that

$$\begin{aligned}\|\partial^\alpha f_j\|_{L^p} &\leq C_\alpha(f, \|u\|_{L^\infty}) 2^{j|\alpha|} \|\Delta_j u\|_{L^p} \\ &\leq c_j C_\alpha(f, \|u\|_{L^\infty}) 2^{-j(s-|\alpha|)} \|u\|_{B^s_{p,r}} \end{aligned} \tag{7.27}$$

with $\|(c_j)\|_{\ell^r} = 1$. We apply Lemma 7.4.4 and the theorem is proved provided we check (7.26). In order to do it, let us recall Faa-di-Bruno's formula.

$$\partial^\alpha g(a) = \sum_{\substack{\alpha_1+\cdots+\alpha_p=\alpha \\ |\alpha_j|\geq 1}} \Big(\prod_{k=1}^p \partial^{\alpha_k} a\Big) g^{(p)}(a).$$

From this formula, we infer that

$$\partial^\alpha m_j = \sum_{\substack{\alpha_1+\cdots+\alpha_p=\alpha \\ |\alpha_j|\geq 1}} \int_0^1 \Big(\prod_{k=1}^p \partial^{\alpha_k}(S_j u + t\Delta_j u)\Big) f^{(p+1)}(S_j u + t\Delta_j u) dt.$$

Using Lemma 7.1.1, we get that

$$\begin{aligned}\|\partial^\alpha m_j\|_{L^\infty} &\leq C_\alpha(f) \sum_{\substack{\alpha_1+\cdots+\alpha_p=\alpha \\ |\alpha_j|\geq 1}} \int_0^1 \Big(\prod_{k=1}^p 2^{j|\alpha_k|} \|u\|_{L^\infty}\Big) \\ &\leq C_\alpha(f, \|u\|_{L^\infty}) 2^{j|\alpha|}.\end{aligned}$$

This proves (7.26) and thus the theorem.

7.5 References and remarks

Ce chapitre is to connaître, to l'exception of the sous-section 7.4.2 and of the proof of the thèorème 7.2.3.

Chapter 8

Quasilinear symmetric systems

8.1 Definition and examples

Let us now define what a quasilinear symmetric system is. First a quasilinear system is a system of the form

$$(S)\begin{cases} \partial_t U + \displaystyle\sum_{k=1}^{d} A_k(U)\partial_k U &= 0 \\ U_{|t=0} &= U_0 \end{cases}$$

where $A = (A_k)_{1\leq k\leq d}$ are smooth functions from $\mathbb{R}^N$ into the space of $N \times N$ matrices with real coefficients.

Definition 8.1.1 *The above system (S) is symmetric if and only if for any $k \in \{1, \dots, d\}$ the function A_k takes its value in the space of symmetric $N \times N$ matrices.*

An example of such a system is given by (1.1) page 4.

8.2 The resolution of quasilinear symmetric systems

The purpose of this section is to prove a local wellposedness for quasilinear symmetric systems

$$(S)\begin{cases} \partial_t U + \displaystyle\sum_{k=1}^{d} A_k(U)\partial_k U &= 0 \\ U_{|t=0} &= U_0 \end{cases}$$

The basic theorem is the following

Theorem 8.2.1 *If U_0 belongs to H^s with $s > d/2 + 1$, a positive time T exists such that a unique solution U of (S) exists in*

$$C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$$

Moreover T can be bounded from below by $c\|\nabla U_0\|_{H^{s-1}}$, where c depends only on the family $A = (A_k)_{1\leq k\leq d}$.

Finally, if $T^\star$ denotes the maximal time of existence of such a solution, then $T^\star$ does not depend on s and

$$T^\star < \infty \Longrightarrow \int_0^{T^\star} \|\nabla U(t,\cdot)\|_{L^\infty} dt.$$

For sake of technical ifmplicity, we shall assume that the functions are of the type

$$A_k(U) = A_k^{(0)} + \sum_{j=1}^{N} A_k^j U_j$$

8.2.1 Basic shemes of resolution for quasilinear symmetric systems

In order to prove this theorem, we shall use an iterative scheme. Let us consider the sequence $(U^{(n)})_{n\in\mathbb{N}}$ defined by $U^{(0)} = 0$ and

$$\left\{\begin{array}{rcl} \partial_t U^{(n+1)} + \displaystyle\sum_{k=1}^{d} A_k(U^{(n)})\partial_k U^{(n+1)} & = & 0 \\ U^{(n+1)}_{|t=0} & = & S_{n+1}U_0. \end{array}\right.$$

Theorem 6.2.1 ensures that this sequence is well defined and that $U^{(n)}$ belongs to $C^1(\mathbb{R}; H^s)$ for any s. The proof proceeds in three steps:

- first, we shall prove that for T small enough, the sequence $(U^{(n)})_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T]; H^s)$,
- then we shall prove that for T small enough, the sequence $(U^{(n)})_{n\in\mathbb{N}}$ is a Cauchy one in $L^\infty([0,T]; H^{s'})$ for any $s' < s$,
- finally we shall prove that the limit of this sequence is a solution of (S) and that it belongs to $C([0,T]; H^s)$.

The two main tools of the proof will be Littlewood-Paley theory and paradifferential computationus.

8.2.2 Paralinearization and energy estimates

The main lemma is the following

Lemma 8.2.1 *Let s be a positive real number. Let us consider U and V two functions of $L^\infty([0,T]; H^s)$ the gradient of which is in $L^\infty([0,T]\times\mathbb{R}^d)$. Let us assume that*

$$\partial_t V + \sum_{k=1}^{d} A_k(U)\partial_k V = F.$$

Stating $V_q \stackrel{\text{def}}{=} \Delta_q V$, we have

$$\partial_t V_q + \sum_{k=1}^{d} (S_{q-1}A_k(U))\partial_k V_q = \Delta_q F + R_q(U,V)$$

where $R_q(U,V)$ satisfies

$$\|R_q(U,V)(t)\|_{L^2} \leq Cc_q(t)2^{-qs}\Big(\|\nabla U(t)\|_{L^\infty}\|V(t)\|_{H^s} + \|\nabla V(t)\|_{L^\infty}\|U(t)\|_{H^s}\Big) \tag{8.1}$$

with, as along this section $\displaystyle\sum_q c_q^2(t) = 1$.

First of all, let us write that

$$\partial_t V_q + \Delta_q \sum_{k=1}^{d} A_k(U)\partial_k V = \Delta_q F.$$

The main point of the proof of the lemma is the commutation between a multiplication and the operator Δ_q. Let us point out that the constant part of $A_k(U)$ does not play any role here because it obviously commutes with Δ_q. Thus we have

$$\Delta_q \sum_{k=1}^{d} A_k(U)\partial_k V = A_k^{(0)}\Delta_q V + \sum_{j,k} A_k^j \Delta_q \left(U^j \partial_k V\right).$$

Let us use a ifmplified version of Bony's decomposition used in Section 7.4. Let us write

$$\begin{aligned} U^j\partial_k V &= T_{U^j}\partial_k V + T'_{\partial_k V}U^j \quad \text{with} \\ T_{U^j}\partial_k V &= \sum_{q'} S_{q'-1}U^j \Delta_{q'}\partial_k V \quad \text{and} \\ T'_{\partial_k V}U^j &= \sum_{q'} S_{q'+2}\partial_k V \Delta_{q'} U^j. \end{aligned}$$

As the support of the Fourier transform of $S_{q'-1}U^j\Delta_{q'}\partial_k V$ is included in a ring of the type $2^{q'}\widetilde{\mathcal{C}}$, we have

$$\begin{aligned} \Delta_q \sum_{q'} S_{q'-1}U^j\Delta_{q'}\partial_k V &= \Delta_q \sum_{|q'-q|\leq N_1} S_{q'-1}U^j\Delta_{q'}\partial_k V \\ &= R_q^1(U,V) + \sum_{|q'-q|\leq N_1} S_{q'-1}U^j\Delta_q\Delta_{q'}\partial_k V \\ &= R_q^1(U,V) + R_q^2(U,V) + S_{q-1}U^j\partial_k V_q \quad \text{with} \\ R_q^1(U,V) &\stackrel{\text{def}}{=} \sum_{|q'-q|\leq N_1} \left[\Delta_q, S_{q'-1}U^j\right]\Delta_{q'}\partial_k V \quad \text{and} \\ R_q^2(U,V) &\stackrel{\text{def}}{=} \sum_{|q'-q|\leq N_1} (S_{q'-1}U^j - S_{q-1}U^j)\Delta_q\Delta_{q'}\partial_k V\cdot \end{aligned}$$

The commutation between the operator Δ_q and the equation can be described by the following formula:

$$\begin{aligned} \partial_t V_q + \sum_{k=1}^{d} S_{q-1}A_k(U)\frac{\partial V_q}{\partial x_k} &= \Delta_q F + \sum_{m=1}^{3} R_q^m(U,V) \quad \text{with} \qquad (8.2) \\ R_q^1(U,V) &\stackrel{\text{def}}{=} \sum_{\substack{|q'-q|\leq N_1 \\ j,k}} A_k^j\left[\Delta_q, S_{q'-1}U^j\right]\Delta_{q'}\partial_k V, \\ R_q^2(U,V) &\stackrel{\text{def}}{=} \sum_{\substack{|q'-q|\leq N_1 \\ j,k}} A_k^j(S_{q'-1}U^j - S_{q-1}U^j)\Delta_q\Delta_{q'}\partial_k V \quad \text{and} \\ R_q^3(U,V) &\stackrel{\text{def}}{=} \Delta_q \sum_{j,k} A_k^j T'_{\partial_k V}U^j. \end{aligned}$$

The first term is estiamted thanks to the following lemma.

Lemma 8.2.2 *A constant C exists such that, for any $p \in [1,+\infty]$, any lipschitz function a and any function b in L^p,*

$$\|[\Delta_q, a]b\|_{L^p} \leq C 2^{-q} \|\nabla a\|_{L^\infty} \|b\|_{L^p}.$$

In order to prove this lemma, let us think Δ_q as a convolution and let us write

$$\begin{aligned}\Big([\Delta_q, a]b\Big)(x) &= \Delta_q(ab)(x) - a(x)\Delta_q b(x)\\ &= 2^{qd}\int_{\mathbb{R}^d} h(2^q(x-y))(a(y)-a(x))b(y)dy.\end{aligned}$$

As the function a is supposed to be lipschitz, we have

$$|a(y)-a(x)| \leq \|\nabla a\|_{L^\infty}|y-x|.$$

It turns out that

$$|\Big([\Delta_q, a]b\Big)(x)| \leq 2^{qd}\|\nabla a\|_{L^\infty}\int_{\mathbb{R}^d} |h(2^q(x-y))|\,|y-x|\,|b(y)|dy.$$

Then Young inequality implies that

$$\|[\Delta_q, a]b\|_{L^p} \leq 2^{-q}\|h(\cdot)|\cdot|\,\|_{L^1}\|\nabla a\|_{L^\infty}\|b\|_{L^p}.$$

This concludes the proof of the lemma.

The lemma implies that

$$\|R_q^1(U,V)\|_{L^2} \leq C 2^{-q} \sum_{\substack{|q'-q|\leq N_1\\ 1\leq k\leq d}} \|\nabla S_{q'-1}U\|_{L^\infty}\left\|\Delta_{q'}\partial_k V\right\|_{L^2}.$$

Lemma 7.1.1 page 75 and the fact that $\|S_q a\|_{L^\infty} \leq C\|a\|_{L^\infty}$ imply that

$$\|R_q^1(U,V)\|_{L^2} \leq C \sum_{\substack{|q'-q|\leq N_1\\ 1\leq k\leq d}} 2^{-q-q'(s-1)}\|\nabla U\|_{L^\infty} 2^{q'(s-1)}\left\|\Delta_{q'}\partial_k V\right\|_{L^2}.$$

But as we have $|q'-q| \leq N_1$, we get by definition of the H^s norm

$$\|R_q^1(U,V)\|_{L^2} \leq C c_q 2^{-qs}\|\nabla U\|_{L^\infty}\|\nabla V\|_{H^{s-1}}. \tag{8.3}$$

In order to estimate $R_q^2(U,V)$, let us use again Lemma 7.1.1 page 75 which tells us that

$$\|\Delta_p U\|_{L^\infty} \leq C 2^{-p}\|\nabla U\|_{L^\infty}.$$

This implies that

$$\|R_q^2(U,V)\|_{L^2} \leq C \sum_{\substack{|q'-q|\leq N_1\\ p\in[q-1,q'-1]}} 2^{-p} c_{q'} 2^{-q'(s-1)}\|\nabla U\|_{L^\infty}\|\nabla V\|_{H^{s-1}}.$$

It turns out that

$$\|R_q^2(U,V)\|_{L^2} \leq C c_q 2^{-qs}\|\nabla U\|_{L^\infty}\|\nabla V\|_{H^{s-1}}. \tag{8.4}$$

Theorems 7.4.1 page 85 and 7.4.3 page 86 implies that

$$\|T'_{\partial_k V}U\|_{H^s} \leq C\|\nabla V\|_{L^\infty}\|U\|_{H^s}.$$

By definition of H^s norm, we get that

$$\|R_q^3(U,V)\|_{L^2} \leq C c_q 2^{-qs}\|\nabla V\|_{L^\infty}\|U\|_{H^s}. \tag{8.5}$$

Putting the three estimates (8.3)–(8.5) together implies the lemma.

Let us apply this lemma to achieve the first step of the proof of Theorem 8.2.1. In order to do so, let us prove by induction that for a suitable constant C_0, we have

$$4C_0T\|U_0\|_{H^s} < 1 \Longrightarrow \forall n \in \mathbb{N}\,,\ \|U^{(n)}\|_{L^\infty([0,T];H^s)} \leq 2\|U_0\|_{H^s}. \tag{8.6}$$

The above assertion is of course true for $n = 0$. Let us assume it for some n. Lemma 8.2.1 allows us to write that

$$\partial_t U_q^{(n+1)} + \sum_{k=1}^{d}(S_{q-1}A_k(U^{(n)}))\partial_k U_q^{(n+1)} = R_q(U^{(n)}, U^{(n+1)}).$$

The L^2 energy estimate (6.2) implies that

$$\frac{1}{2}\frac{d}{dt}\|U_q^{(n+1)}(t)\|_{L^2}^2 \leq C\|\nabla U^{(n)}(t)\|_{L^\infty}\|U_q^{(n+1)}(t)\|_{L^2}^2 + C\|R_q(U^{(n)}, U^{(n+1)})\|_{L^2}\|U_q^{(n+1)}(t)\|_{L^2}.$$

Using Lemma 8.2.1, the fact that as $s - 1 > \dfrac{d}{2}$, the space H^{s-1} is continuously embedded in L^∞ and the induction hypothesis, we get, for any $t \leq T$,

$$\frac{d}{dt}\|U_q^{(n+1)}(t)\|_{L^2}^2 \leq C\|U_0\|_{H^s}\|U_q^{(n+1)}(t)\|_{L^2}\Big(\|U_q^{(n+1)}(t)\|_{L^2} + c_q(t)2^{-qs}\|U_q^{(n+1)}(t)\|_{L^2}\Big).$$

By definition of the Sobolev norm, we get

$$\frac{d}{dt}\|U_q^{(n+1)}(t)\|_{L^2}^2 \leq C\|U_0\|_{H^s}c_q^2(t)2^{-2qs}\|U^{(n+1)}(t)\|_{H^s}^2.$$

By time integration, we obtain that

$$\|U_q^{(n+1)}\|_{L^\infty([0,T];L^2)}^2 \leq \|\Delta_q U_0\|_{H^s}^2 + 2C\|U_0\|_{H^s}\|U^{(n+1)}\|_{L^\infty([0,T];H^s)}^2 2^{-2qs}\int_0^T c_q^2(t)dt.$$

Let us remine that for any t, we have $\displaystyle\sum_q c_q^2(t) = 1$. Multipliying by 2^{2qs} and taking the sum over q gives,

$$\begin{aligned}\sum_q 2^{2qs}\|U_q^{(n+1)}\|_{L^\infty([0,T];L^2)}^2 &\leq \|U_0\|_{H^s}^2 + 2C\|U_0\|_{H^s}\|U^{(n+1)}\|_{L^\infty([0,T];H^s)}^2\sum_q\int_0^T c_q^2(t)dt \\ &\leq \|U_0\|_{H^s}^2 + 2C\|U_0\|_{H^s}T\|U^{(n+1)}\|_{L^\infty([0,T];H^s)}^2. \end{aligned}\tag{8.7}$$

Let us observe that

$$\|V\|_{L^\infty([0,T];H^s)}^2 \leq \sum_q 2^{2qs}\|\Delta_q V\|_{L^\infty([0,T];L^2)}^2$$

Then choosing $C_0 \geq 4C$ where C is the constant that appears in the above inequality we get that

$$\|U^{(n+1)}\|_{L^\infty([0,T];H^s)} \leq 2\|U_0\|_{H^s}. \tag{8.8}$$

This is the conclusion of the first step of the proof.

Remark Let us point out that we proved a little bit more than what we announced. In fact plugging (8.8) into (8.7) gives

$$\sum_q 2^{2qs}\|U_q^{(n+1)}\|_{L^\infty([0,T];L^2)}^2 \leq 2\|U_0\|_{H^s}. \tag{8.9}$$

This will be important in the proof of the continuity of the solution with value in H^s.

The second step consists mainely in the proof of the fact that $(U^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(([0,T];L^2)$. Let us assume this. Then using interpolation Inequality (**??**) page **??** and remembering (8.8), we get for any $s'\in]0,s[$,

$$\|U^{(n+p)}-U^{(n)}\|_{L^\infty([0,T];H^{s'})}\leq \|U^{(n+p)}-U^{(n)}\|_{L^\infty([0,T];L^2)}^{1-\frac{s'}{s}}(2\|U_0\|_{H^s})^{\frac{s'}{s}}.$$

By difference, we get

$$\partial_t(U^{(n+1)}(t)-U^{(n)}(t))+\sum_{k=1}^d A_k(U^{(n)})\partial_k(U^{(n+1)}(t)-U^{(n)}(t))$$
$$=-\sum_{k=1}^d\Big(A_k(U^{(n)})-A_k(U^{(n-1)}\Big)\partial_k U^{(n)}.$$

Then using the energy estimate (6.2), we get

$$\frac{d}{dt}\|U^{(n+1)}(t)-U^{(n)}(t)\|_{L^2}^2\leq C\|\nabla U^{(n)}\|_{L^\infty}\|U^{(n+1)}(t)-U^{(n)}(t)\|_{L^2}$$
$$\times\Big(\|U^{(n+1)}(t)-U^{(n)}(t)\|_{L^2}+\|U^{(n-1)}(t)-U^{(n)}(t)\|_{L^2}\Big).$$

Stating $v_{n+1}\stackrel{\text{def}}{=}\|U^{(n+1)}(t)-U^{(n)}(t)\|_{L^\infty([0,T];L^2)}$ and using estimate (8.8) together with the Sobolev embedding $H^{s-1}\subset L^\infty$, we get by integration that

$$v_{n+1}^2\leq\|\Delta_n U_0\|_{L^2}^2+C\|U_0\|_{H^s}T(v_{n+1}^2+v_{n+1}v_n)$$

Let us assume that $2CT\|U_0\|_{H^s}\leq 1$. Then we have

$$v_{n+1}^2\leq 2\|\Delta_n U_0\|_{L^2}^2+v_{n+1}v_n.$$

As $\|\Delta_n U_0\|_{L^2}^2\leq C2^{-2ns}$, the series $(v_n)_{n\in\mathbb{N}}$ converges. This proves that $(U^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(([0,T];L^2)$.

As this implies that it is a Cauchy sequence in $L^\infty(([0,T];H^{s'})$, using the fact that the product maps continuously $H^{s'}\times H^{s'-1}$ into $H^{s'-1}$ when s' is greater than $d/2$, we get that the limit U of $(U^{(n)})_{n\in\mathbb{N}}$ is a solution of (S).

To sum up, the whole existence part of Theorem 8.2.1 is proved except the fact that U is continuous in time with value in H^s. The last step consists in proving it. Unfortunately, the improvement given in (8.9) cannot be used directly. But an analogous inequality will give us the result.

Let us consider a solution U of (S) belonging to

$$L^\infty([0,T];H^s)\cap C([0,T];H^1)\cap C^1([0,T];L^2).$$

We shall prove that U is continuous in time with value in H^s. Let us apply Lemma 8.2.1. We get that $\Delta_q U$ satisfies

$$\begin{cases}\partial_t\Delta_q U+\displaystyle\sum_{k=1}^d(S_{q-1}A_k(U))\partial_k\Delta_q U &= R_q(U,U)\\ \Delta_q U_{|t=0} &= \Delta_q U_0\end{cases}$$

with
$$\|R_q(U,U)(t)\|_{L^2} \leq Cc_q(t)2^{-qs}\|\nabla U(t)\|_{L^\infty}\|U(t)\|_{H^s}.$$
By L^2 energy estimate we get that
$$\frac{d}{dt}\|\Delta_q U(t)\|_{L^2}^2 \leq Cc_q(t)2^{-2qs}\|\nabla U(t)\|_{L^\infty}\|U(t)\|_{H^s}^2. \tag{8.10}$$
By integration, we get
$$\|\Delta_q U(t)\|_{L^2}^2 \leq \|\Delta_q U(0)\|_{L^2}^2 + C2^{-2qs}\int_0^t c_q^2(t')\|\nabla U(t')\|_{L^\infty}\|U(t')\|_{H^s}^2 dt'. \tag{8.11}$$
Using the fact that U belongs to $L^\infty([0,T];H^s)$, we get that
$$\|\Delta_q U\|_{L^\infty([0,T];L^2)}^2 \leq \|\Delta_q U(0)\|_{L^2}^2 + C2^{-2qs}\|U\|_{L^\infty([0,T];H^s)}^3\Big(\int_0^T c_q^2(t)dt\Big).$$
After multiplication by 2^{2qs} and summation in q, it turns out that
$$\sum_q 2^{2qs}\|\Delta_q U\|_{L^\infty([0,T];L^2)}^2 \leq \|U_0\|_{H^s}^2 + CT\|U\|_{L^\infty([0,T];H^s)}^3.$$
Now let us consider any positive ε. The above inequality implies that an integer N exists such that
$$\sum_{q\geq N} 2^{2qs}\|\Delta_q U\|_{L^\infty([0,T];L^2)}^2 \leq \frac{\varepsilon^2}{4}\cdot$$
Thus we have
$$\begin{aligned}\|U(t)-U(t')\|_{H^s}^2 &\leq \sum_{q<N} 2^{2qs}\|\Delta_q(U(t)-U(t'))\|_{L^2}^2 \\ &\qquad + 2\sum_{q\geq N} 2^{2qs}\|\Delta_q U\|_{L^\infty([0,T];L^2)}^2 \\ &\leq \sum_{q<N} 2^{2qs}\|\Delta_q(U(t)-U(t'))\|_{L^2}^2 + \frac{\varepsilon^2}{2} \\ &\leq C2^{Ns}\|U(t)-U(t')\|_{L^2}^2 + \frac{\varepsilon^2}{2}\cdot\end{aligned}$$
Thus $U \in C([0,T];H^s)$.

The uniqueness is an obvious consequence of the following proposition.

Proposition 8.2.1 *Let U and V be two solutions of (S) in $C([0,T];H^1)\cap C^1(([0,T];L^2)$ the gradient of which is continuous and bounded on $[0,T]\times\mathbb{R}^d$. Then we have*
$$\|U(t)-V(t)\|_{L^2} \leq \|U_0-V_0\|_{L^2}\exp\Big(C\int_0^t(\|\nabla U(t')\|_{L^\infty}+\|\nabla V(t')\|_{L^\infty})dt'\Big)$$

By difference, we get
$$\partial_t(U(t)-V(t)) + \sum_{k=1}^d A_k(U)\partial_k(U-V) = \sum_{k=1}^d A_k(V-U)\partial_k V\cdot$$
Using (6.3) which is valid under the assumptions of the proposition, we get the result.

In order to prove the blow up condition, let us first observe that the maximal time of existence $T^\star$ satisfies

$$T^\star \geq c\|U_0\|_{H^s}.$$

Of course, the maximal time of existence of the solution of (S) with initial data $U(t)$ is $T^\star - t$; Thus we have

$$T^\star - t \geq c\|U(t)\|_{H^s},$$

which can be written

$$\|U(t)\|_{H^s} \geq \frac{C}{(T^\star - t)} \cdot \tag{8.12}$$

This implies that $\|U(t)\|_{H^s}$ does not remain bounded when t tends to $T^\star$. Now let us apply (8.11) and multiply this inequality by 2^{2qs}. After summation, we get

$$\|U(t)\|^2_{H^s} \leq \|U(0)\|^2_{H^s} + C\int_0^t \|\nabla U(t')\|_{L^\infty}\|U(t')\|^2_{H^s}dt'. \tag{8.13}$$

Gronwall's Lemma implies that, for any $t < T^\star$,

$$\|U(t)\|^2_{H^s} \leq \|U(0)\|^2_{H^s} \exp\Big(C\int_0^t \|\nabla U(t')\|_{L^\infty}dt'\Big).$$

Thanks to Inequality (8.12), the whole Theorem 8.2.1 is proved.

8.3 References and Remarks

The proofs of this chapitre ne are pas to savoir.