Problem 1. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}.$$

(Vo Quoc Ba Can)

SOLUTION. Notice that if $a \geq b \geq c$ then

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)-\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)=\frac{(a-b)(a-c)(c-b)}{abc} \leq 0$$

so it's enough to consider one case $a \geq b \geq c$. By squaring two sides, we get

$$\sum_{cyc}\frac{a^2}{b^2}+\sum_{cyc}\frac{2b}{a} \geq \frac{9(a^2+b^2+c^2)}{ab+bc+ca}.$$

Moreover, paying attention to following identities

$$\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-3=\frac{(b-c)^2}{bc}+\frac{(a-b)(a-c)}{ac}$$

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-3=\frac{(b-c)^2(b+c)^2}{b^2c^2}+\frac{(a^2-b^2)(a^2-c^2)}{a^2b^2}$$

and $a^2+b^2+c^2-(ab+bc+ca)=(b-c)^2+(a-b)(a-c)$, so we can rewrite this inequality to $(b-c)^2M+(a-b)(a-c)N \geq 0$ with

$$M=\frac{2}{bc}+\frac{(b+c)^2}{b^2c^2}-\frac{9}{ab+bc+ca} \quad ;$$

$$N=\frac{2}{ac}+\frac{(a+b)(a+c)}{a^2b^2}-\frac{9}{ab+bc+ca} \quad ;$$

If $b-c \geq a-b$ then $2(b-c)^2 \geq (a-b)(a-c)$. Certainly, we have

$$M \geq \frac{6}{bc}-\frac{9}{ab+bc+ca} \geq 0 \ ;$$

$$M+2N \geq \frac{6}{bc}-\frac{18}{ab+bc+ca} \geq 0 \ ;$$

So we can conclude that

$$M(b-c)^2+N(a-b)(a-c) \geq \frac{1}{2}(a-b)(a-c)(M+2N) \geq 0.$$

Now suppose that $b-c \leq a-b$, then $3b \leq a+c$. Certainly $M \geq 0$ and

$$N \geq \frac{2}{ac}+\frac{a+b+c}{ab^2} \geq \frac{2}{ac}+\frac{3}{ab} \geq \frac{(\sqrt{2}+\sqrt{3})^2}{ac+ab} > \frac{9}{ab+bc+ca}.$$

This ends the proof. The equality holds for $a=b=c$.