Worst-case response time analysis of real-time tasks under fixed-priority scheduling with deferred preemption revisited
– with extensions for ECRTS'07 –[*]

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Abstract

Fixed-priority scheduling with deferred preemption (FPDS) has been proposed in the literature as a viable alternative to fixed-priority pre-emptive scheduling (FPPS), that obviates the need for non-trivial resource access protocols and reduces the cost of arbitrary preemptions.

This paper shows that existing worst-case response time analysis of hard real-time tasks under FPDS, arbitrary phasing and relative deadlines at most equal to periods is pessimistic and/or optimistic. The same problem also arises for fixed-priority non-pre-emptive scheduling (FPNS), being a special case of FPDS. This paper provides a revised analysis, resolving the problems with the existing approaches. The analysis is based on known concepts of critical instant and busy period for FPPS. To accommodate for our scheduling model for FPDS, we need to slightly modify existing definitions of these concepts. The analysis assumes a continuous scheduling model, which is based on a partitioning of the timeline in a set of non-empty, right semi-open intervals. It is shown that the critical instant, longest busy period, and worst-case response time for a task are suprema rather than maxima for all tasks, except for the lowest priority task, i.e. that instant, period, and response time cannot be assumed. Moreover, it is shown that the analysis is not uniform for all tasks, i.e. the analysis for the lowest priority task differs from the analysis of the other tasks. These anomalies for the lowest priority task are an immediate consequence of the fact that only the lowest priority task cannot be blocked. To build on earlier work, the worst-case response time analysis for FPDS is expressed in terms of known worst-case analysis results for FPPS. The paper includes pessimistic variants of the analysis, which are uniform for all tasks.

1 Introduction

1.1 Motivation

Based on the seminal paper of Liu and Layland [29], many results have been achieved in the area of analysis for fixed-priority preemptive scheduling (FPPS). Arbitrary preemption of real-time tasks has a number of drawbacks, though. In systems requiring mutual access to shared resources, arbitrary preemptions induce the need for non-trivial resource access protocols, such as the priority ceiling protocol [34]. In systems using cache memory, e.g. to bridge the speed gap between processors and main memory, arbitrary preemptions induce additional cache flushes and reloads. As a consequence, system performance and predictability are degraded, complicating system design, analysis and testing [15, 20, 26, 31, 35]. Although fixed-priority non-preemptive scheduling (FPNS) may resolve these problems, it generally leads to reduced schedulability compared to FPPS. Therefore, alternative scheduling schemes have been proposed between the extremes of arbitrary preemption and no preemption. These schemes are also known as deferred preemption or co-operative scheduling [12], and are denoted by fixed-priority scheduling with deferred preemption (FPDS) in the remainder of this paper.

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[*] This document is an extension of [10], addressing the comment of the reviewers of the Euromicro Conference of Real-Time Systems 2007 (ECRTS'07) on a paper derived from [10].

Worst-case response time analysis of periodic real-time tasks under FPDS, arbitrary phasing, and relative deadlines within periods has been addressed in a number of papers [11, 12, 15, 26]. The existing analysis is not exact, however. In [11], it has already been shown that the analysis presented in [12, 15, 26] is pessimistic. More recently, it has been shown in [6, 7] that the analysis presented in [11, 12, 15] is optimistic. Unlike the implicit assumptions in those latter papers, the worst-case response time of a task under FPDS and arbitrary phasing is not necessarily assumed for the first job of that task upon its critical instant. Hence, the existing analysis may provide guarantees for tasks that in fact miss their deadlines in the worst-case. In [8, 9], it has been shown that the latter problem also arises for FPNS, being a special case of FPDS, and its application for the schedulability analysis of controller area networks (CAN) [37, 38, 39]. Revised analysis for CAN resolving the problem with the original approach in an evolutionary fashion can be found in [17].

1.2 Contributions

This paper resolves the problems with the existing approaches by presenting a novel worst-case response time analysis for hard real-time tasks under FPDS, arbitrary phasing and arbitrary relative deadlines. The analysis assumes a *continuous* scheduling model rather than a *discrete* scheduling model [4], e.g. all task parameters are taken from the real numbers. The motivation for this assumption stems from the observation that a discrete view on time is in many situations insufficient; see for example [2, 22, 25]. The scheduling model is based on a partitioning of the timeline in a set of non-empty, right semi-open intervals [16, 22]. The analysis is based on the concepts of *critical instant* [29] and *busy period* [27]. To accommodate for our scheduling model for FPDS, we need to slightly modify the existing definitions of these concepts. To prevent confusion with the existing definition of busy period, we use the term *active period* for our definition in this document.

In this document, we discuss conditions for termination of an active period, and present a sufficient condition with a formal proof. Moreover, we show that the critical instant, longest active period, and worst-case response time for a task are suprema rather than maxima for all tasks, except for the lowest priority task, i.e. that instant, period, and response time cannot be assumed. Our worst-case response time analysis is not uniform for all tasks. In particular, the analysis for the lowest priority task differs from the analysis for the other tasks. These anomalies for the lowest priority task are an immediate consequence of the fact that, unlike the other tasks, the lowest priority task cannot be blocked. To build on earlier results, worst-case response times under FPDS are expressed in terms of worst-case response times and worst-case occupied times [5] under FPPS. We also present pessimistic variants of the analysis, which are indeed uniform for all tasks, and show that the revised analysis for CAN presented in [17] conforms to a pessimistic variant.

1.3 Related work

Next to continuous scheduling models, one can find discrete scheduling models in the literature, e.g. in [18, 21], and models in which domains are not explicitly specified [16, 24, 30]. Because the equations for response time analysis depend on the model, we prefer to be explicit about the domains in our model. As mentioned above, our scheduling model is based on a partitioning of the timeline in a set of non-empty, right semi-open intervals. Alternatively, the scheduling model in [30] is based on *left* semi-open intervals.

In this paper, we assume that each job (or activation) of a task consists of a sequence of non-preemptable subjobs, where each subjob has a known worst-case computation time, and present novel worst-case response time analysis to determine schedulability of tasks under FPDS. Similarly, George et al. assume in [18] that the worst-case computation time of each non-preemptive job is known, and present worst-case response time analysis of tasks under FPNS. Conversely, Baruah [3] determines the largest non-preemptive ‘chunks’ into which jobs of a task can be broken up to still ensure feasibility under earliest deadline first (EDF).

For worst-case response time analysis of tasks under FPPS, arbitrary phasing, and relative deadlines at most equal to periods, it suffices to determine the response time of the first job of a task upon its critical instant. For tasks with relative deadlines larger than their respective periods, Lehoczky [27] introduced the concept of a busy period, and showed that all jobs of a task in a busy period need to be considered to determine its worst-case response time. Hence, when the relative deadline of a task is larger than its period, the worst-case response time of that task is not necessarily assumed for the first job of a task when released at a critical instant. Similarly, González Harbour et al. [19] showed that if relative deadlines are at most equal to periods, but priorities vary during execution, then again multiple jobs must be considered to determine the worst-case response time. Initial work on pre-emption thresholds [40] failed to identify this issue. The resulting flaw was later corrected by Regehr [33]. Worst-case response time analysis of tasks under EDF and relative deadlines at most equal to periods described by Spuri [36] is also based on the concept of busy period.

1.4 Structure

This paper has the following structure. First, in Section 2, we present basic real-time scheduling models for FPPS and FPDS. Next, worst-case analysis for FPPS is briefly recapitulated in Section 3. Section 4 presents various examples refuting the existing worst-case response time analysis for FPDS. The notion of active period is the topic of Section 5. We present a formal definition of active period and theorems with a recursive equation for the length of an active period and an iterative procedure to determine its value. Worst-case analysis for FPDS is addressed in Section 6. We present a theorem for critical instant and theorems to determine the worst-case response time of a task under FPDS and arbitrary phasing. Section 7 illustrates the worst-case response time analysis by applying it to some examples presented in Section 4. Section 8 compares the notion of level-*i* active period with similar definitions in the literature, presents pessimistic variants of the worst-case response time analysis, and illustrates the revised analysis for an advanced model for FPDS. The paper is concluded in Section 9.

2 Real-time scheduling models

This section starts with a presentation of a basic real-time scheduling model for FPPS. Next, that basic model is refined for FPDS. The section is concluded with remarks.

2.1 Basic model for FPPS

We assume a single processor and a set $\mathcal{T}$ of n periodically released, independent tasks $\tau_1, \tau_2, \ldots, \tau_n$ with unique, fixed priorities. At any moment in time, the processor is used to execute the highest priority task that has work pending. So, when a task τ_i is being executed, and a release occurs for a higher priority task τ_j, then the execution of τ_i is preempted, and will resume when the execution of τ_j has ended, as well as all other releases of tasks with a higher priority than τ_i that have taken place in the meantime.

A *schedule* is an assignment of the tasks to the processor. A schedule can be defined as an integer step function $\sigma : \mathbb{R} \rightarrow \{0, 1, \ldots, n\}$, where $\sigma(t) = i$ with $i > 0$ means that task τ_i is being executed at time t, while $\sigma(t) = 0$ means that the processor is idle. More specifically, we define $\sigma(t)$ as a right-continuous and piece-wise constant function, i.e. σ partitions the timeline in a set of non-empty, right semi-open intervals $\{[t_j, t_{j+1})\}_{j \in \mathbb{Z}}$. At times t_j, the processor performs a *context switch*. Figure 1 shows an example of the execution of a set $\mathcal{T}$ of three periodic tasks and the corresponding value of the schedule $\sigma(t)$.

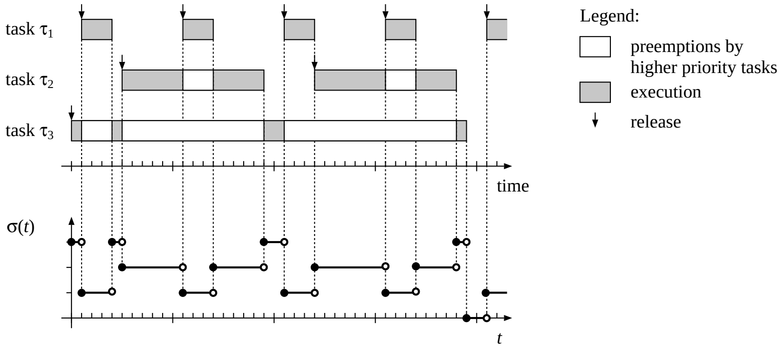

Figure 1. An example of the execution of a set $\mathcal{T}$ of three independent periodic tasks τ_1, τ_2, and τ_3, where task τ_1 has highest priority, and task τ_3 has lowest priority, and the corresponding value of $\sigma(t)$.

Each task τ_i is characterized by a (*release*) *period* $T_i \in \mathbb{R}^+$, a *computation time* $C_i \in \mathbb{R}^+$, a (*relative*) *deadline* $D_i \in \mathbb{R}^+$, where $C_i \leq \min(D_i, T_i)$, and a *phasing* $\varphi_i \in \mathbb{R}^+ \cup \{0\}$. An *activation* (or *release*) *time* is a time at which a task τ_i becomes ready for execution. A release of a task is also termed a *job*. The first job of task τ_i is released at time φ_i and is referred to as job zero. The release of job k of τ_i therefore takes place at time $a_{ik} = \varphi_i + kT_i$, $k \in \mathbb{N}$. The (*absolute*) *deadline* of job k of τ_i takes place at $d_{ik} = a_{ik} + D_i$. The *begin* (or *start*) *time* b_{ik} and *finalization* (or *completion*) *time* f_{ik} of job k of τ_i is the time at which τ_i actually starts and ends the execution of that job, respectively. The set of phasings φ_i is termed the phasing φ of the task set $\mathcal{T}$.

The *active* (or *response*) *interval* of job k of τ_i is defined as the time span between the activation time of that job and its finalization time, i.e. $[a_{ik}, f_{ik})$. The *response time* r_{ik} of job k of τ_i is defined as the length of its active interval, i.e. $r_{ik} = f_{ik} - a_{ik}$. Figure 2 illustrates the above basic notions for an example job of task τ_i.

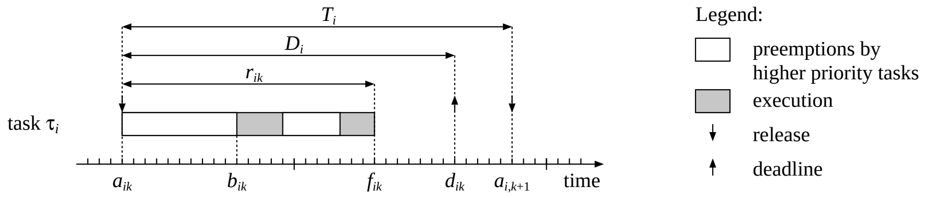

Figure 2. Basic model for task τ_i.

The *worst-case response time* WR_i of a task τ_i is the largest response time of any of its jobs, i.e.

$$WR_i = \sup_{\varphi,k} r_{ik}. \tag{1}$$

In many cases, we are not interested in the worst-case response time of a task for a particular computation time, but in the value as a function of the computation time. We will therefore use a functional notation when needed, e.g. $WR_i(C_i)$. A *critical instant* of a task is defined to be an (hypothetical) instant that leads to the worst-case response time for that task. Typically, such an instant is described as a point in time with particular properties. As an example, a critical instant for tasks under FPPS is given by a point in time for which all tasks have a simultaneous release.

We assume that we do not have control over the phasing φ, for instance since the tasks are released by external events, so we assume that any arbitrary phasing may occur. This assumption is common in real-time scheduling literature [23, 24, 29]. We also assume other standard basic assumptions [29], i.e. tasks are ready to run at the start of each period and do no suspend themselves, tasks will be preempted instantaneously when a higher priority task becomes ready to run, a job of task τ_i does not start before its previous job is completed, and the overhead of context switching and task scheduling is ignored. Finally, we assume that the deadlines are hard, i.e. each job of a task must be completed at or before its deadline. Hence, a set $\mathcal{T}$ of n periodic tasks can be scheduled if and only if

$$WR_i \leq D_i \tag{2}$$

for all $i = 1,\ldots,n$. For notational convenience, we assume that the tasks are given in order of decreasing priority, i.e. task τ_1 has highest priority and task τ_n has lowest priority.

The (*processor*) *utilization factor* U is the fraction of the processor time spent on the execution of the task set [29]. The fraction of processor time spent on executing task τ_i is C_i/T_i, and is termed the *utilization factor* U_i^τ of task τ_i, i.e.

$$U_i^\tau = \frac{C_i}{T_i}. \tag{3}$$

The *cumulative utilization factor* U_i for tasks τ_1 till τ_i is the fraction of processor time spent on executing these tasks, and is given by

$$U_i = \sum_{j \leq i} U_j^\tau. \tag{4}$$

Therefore, U is equal to the cumulative utilization factor U_n for n tasks.

$$U = U_n = \sum_{j \leq n} U_j^\tau = \sum_{j \leq n} \frac{C_j}{T_j}. \tag{5}$$

In [29], the following necessary condition is determined for the schedulability of a set $\mathcal{T}$ of n periodic tasks under any scheduling algorithm.

$$U \leq 1. \tag{6}$$

Unless explicitly stated otherwise, we assume in this document that task sets satisfy this condition.

2.2 Refined model for FPDS

For FPDS, we need to refine our basic model of Section 2.1. Each job of task τ_i is now assumed to consist of a sequence of m_i subjobs. The k^{th} subjob of τ_i is characterized by a computation time $C_{ik} \in \mathbb{R}^+$, where $C_i = \sum_{k=1}^{m_i} C_{ik}$. We assume that subjobs are non-preemptable. Hence, tasks can only be preempted at subjob boundaries, i.e. at so-called *preemption points*. For convenience, we will use the term F_i to denote the computation time C_{i,m_i} of the final subjob of τ_i. Note that when $m_i = 1$ for all i, we have FPNS as special case.

2.3 Concluding remarks

In this document, we will use the superscript P to denote FPPS, e.g. WR_i^{P} denotes the worst-case response time of task τ_i under FPPS and arbitrary phasing. Similarly, we will use the superscripts D and N to denote FPDS and FPNS, respectively.

In our basic model for FPPS, we introduced notions for points in time with a subscript identifying a task and optionally a job of that task, e.g. a_{ik} is the release time of job k of task τ_i. In this document, we will need similar notions that are expressed relative to a particular moment in time, e.g. the relative release time of the first job of a task at or after time t_s. We will therefore also use relative versions of the notions, where relative can refer to the identification of the job and/or to the particular moment in time, depending on the notion. As an example, let $\phi_i(t)$ denote the earliest (absolute) activation of a job of task τ_i at or after time t, i.e.

$$\phi_i(t) = \varphi_i + \left(\left\lceil \frac{t - \varphi_i}{T_i} \right\rceil \right)^+ \cdot T_i.$$

Here, the notation x^+ stands for $\max(x,0)$, which is used to indicate that there are no releases of τ_i before time φ_i. Because $\varphi_i \geq 0$, the term $\left(\left\lceil \frac{t-\varphi_i}{T_i} \right\rceil \right)^+$ is equal to the number of releases of τ_i in $[0,t)$. Given $\phi_i(t)$, the *relative phasing* $\varphi_i(t)$ is given by $\varphi_i(t) = \phi_i(t) - t$. The release of job k of task τ_i relative to t takes place at the *relative activation time* $a_{ik}(t) = \varphi_i(t) + kT_i$, $k \in \mathbb{N}$. For $a_{ik}(t)$, both the identification of the job and the time are therefore relative to t. Similarly, the notions *relative begin time* $b_{ik}(t)$ and *relative finalization time* $f_{ik}(t)$ denote a time relative to t and concern the job k of task τ_i relative to t. For the *relative response time* $r_{ik}(t)$, only the identification of the job is relative to t. We will use abbreviated representations for the relative notions using a prime ($'$) when the particular moment in time is clear from the context. As an example, in a context concerning a particular moment t_s, the relative activation time a'_{ik} denotes $a_{ik}(t_s)$.

3 Recapitulation of worst-case analysis for FPPS

For the analysis under FPPS, we only consider cases where the deadlines of tasks are less than or equal to the respective periods. For illustration purposes, we will use a set $\mathcal{T}_1$ of two independent periodic tasks τ_1 and τ_2 with characteristics as given in Table 1.

	$T_i = D_i$	C_i
τ_1	5	2
τ_2	7	3

Table 1. Task characteristics of $\mathcal{T}_1$.

Figure 3 shows an example of the execution of the tasks τ_1 and τ_2 under FPPS. Note that even an infinitesimal increase of the computation time of either task τ_1 or τ_2 will immediately cause the job of task τ_2 released at time 0 to miss its deadline at time 7.

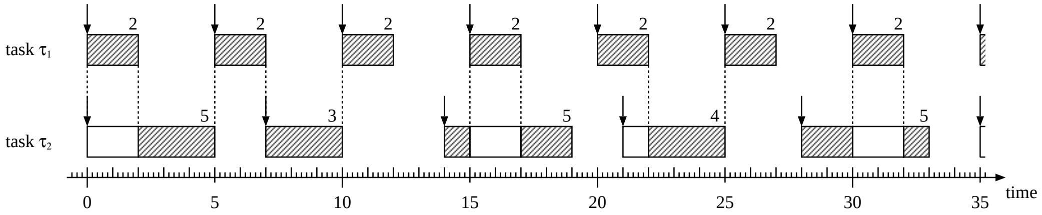

Figure 3. Timeline for $\mathcal{T}_1$ under FPPS with a simultaneous release of both tasks at time zero. The numbers to the top right corner of the boxes denote the response times of the respective releases.

3.1 Worst-case response times

This section presents theorems for the notion of critical instant and to determine worst-case response times of tasks. Although these theorems are taken from [5], most of these results were already known; see for example [1, 23, 29]. Auxiliary lemmas on which the proofs of these theorems and theorems in subsequent sections are based are included in Appendix A.

Theorem 1 (Theorem 4.1 in [5]). *In order to have a maximal response time for an execution k of task* τ_i*, i.e. to have* $f_{ik} - a_{ik} = WR_i$*, we may assume without loss of generality that the phasing* φ *is such that* $\varphi_j = a_{ik}$ *for all* $j < i$*. In other words, the phasing of the tasks' release times is such that the release of the considered execution of* τ_i *coincides with the simultaneous release for all higher priority tasks. This latter point in time is called a* critical instant *for task* τ_i*.* □

Given this theorem, we conclude that time 0 in Figure 3 is a critical instant for both task τ_1 and τ_2. From this figure, we therefore derive that the worst-case response times of tasks τ_1 and τ_2 are 2 and 5, respectively. The next theorems can be used to determine the worst-case response times analytically.

Theorem 2 (Theorem 4.2 in [5]). *The worst-case response time WR_i of a task τ_i is given by the smallest $x \in \mathbb{R}^+$ that satisfies the following equation, provided that x is at most T_i.*

$$x = C_i + \sum_{j<i} \left\lceil \frac{x}{T_j} \right\rceil C_j \tag{7}$$

□

Theorem 3 (Theorem 4.3 in [5]). *The worst-case response time WR_i of task τ_i can be found by the following iterative procedure.*

$$WR_i^{(0)} = C_i \tag{8}$$

$$WR_i^{(l+1)} = C_i + \sum_{j<i} \left\lceil \frac{WR_i^{(l)}}{T_j} \right\rceil C_j, \quad l = 0,1,\ldots \tag{9}$$

The procedure is stopped when the same value is found for two successive iterations of l, or when the deadline D_i is exceeded.

□

3.2 Worst-case occupied times

In Figure 3, task τ_2 is preempted at time 15 due to a release of task τ_1, and resumes its execution at time 17. The span of time from a task τ's release till the moment in time that τ can start or resume its execution after completion of a computation time C is termed *occupied time*. The *worst-case occupied time* (WO) of a task τ is the longest possible span of time from a release of τ till the moment in time that τ can start or resume its execution after completion of a computation C. In [5], it has been shown that the worst-case occupied time can be described in terms of the worst-case response time as follows.

$$WO_i(C_i) = \lim_{x \downarrow C_i} WR_i(x). \tag{10}$$

Considering Figure 3, we derive that worst-case occupied times $WO_2(0)$ and $WO_2(C_2)$ of task τ_2 are equal to 2 and 7, respectively. The next theorems can be used to determine the worst-case occupied times analytically.

Theorem 4 (Theorem 4.4 in [5]). *When the smallest positive solution of (7) for a computation time C_i' is at most D_i, the worst-case occupied time WO_i of a task τ_i with a computation time $C_i \in [0, C_i']$ is given by the smallest non-negative $x \in \mathbb{R}$ that satisfies*

$$x = C_i + \sum_{j<i} \left(\left\lfloor \frac{x}{T_j} \right\rfloor + 1 \right) C_j. \tag{11}$$

□

Theorem 5 (Theorem 4.5 in [5]). *The worst-case occupied time WO_i of task τ_i can be found by the following iterative procedure.*

$$WO_i^{(0)} = \begin{cases} \sum_{j<i} C_j & \text{for } C_i = 0 \\ WR_i & \text{for } C_i > 0 \end{cases} \tag{12}$$

$$WO_i^{(l+1)} = C_i + \sum_{j<i} \left(\left\lfloor \frac{WO_i^{(l)}}{T_j} \right\rfloor + 1 \right) C_j, \quad l = 0,1,\ldots \tag{13}$$

The procedure is stopped when the same value is found for two successive iterations of l. □

3.3 Concluding remarks

The proof of Theorem 4 derives Equation (11) by starting from Equation (10) and subsequently using Lemma 16.

Similarly to Equation 10, we can express WR_i in terms of WO_i, i.e.

$$WR_i(C_i) = \lim_{x \uparrow C_i} WO_i(x). \tag{14}$$

The next two equations express that $WR_i(C_i)$ and $WO_i(C_i)$ are left-continuous and right-continuous, respectively.

$$WR_i(C_i) = \lim_{x \uparrow C_i} WR_i(x) \tag{15}$$

$$WO_i(C_i) = \lim_{x \downarrow C_i} WO_i(x) \tag{16}$$

Lemmas related to these latter three equations can be found in Appendix A.

4 Existing response time analysis for FPDS refuted

In this section, we first recapitulate existing response time analysis under FPDS. Next, we show that the existing analysis is pessimistic. We subsequently give examples refuting the analysis, i.e. examples that show that the existing analysis is optimistic.

4.1 Recapitulation of existing worst-case response time analysis for FPDS

In this section, we recapitulate existing worst-case response time analysis for FPDS with arbitrary phasing and deadlines within periods as described in [12, 15]. We include a recapitulation of the analysis for FPNS as presented in [39]. The main reason for including the latter is that it looks different from the analysis for FPDS and is a basis for the analysis of controller area network (CAN).

4.1.1 Existing analysis for FPDS

The non-preemptive nature of subjobs may cause blocking of a task by at most one lower priority task under FPDS. Moreover, a task can be blocked by at most one subjob of a lower priority task. The maximum blocking B_i^{D} of task τ_i by a lower priority task is therefore equal to the longest computation time of any subjob of a task with a priority lower than task τ_i. This blocking time is given by

$$B_i^{\mathrm{D}} = \max_{j>i} \max_{1 \le k \le m_j} C_{j,k}. \tag{17}$$

Strictly spoken, this blocking time is a supremum (and not a maximum) for all tasks, except for the lowest priority task, i.e. that value cannot be assumed for $i < n$.

The worst-case response time $\widetilde{WR}_i^{\mathrm{D}}$ of a task τ_i under FPDS, arbitrary phasing, and deadlines less than or equal to periods, as presented in [12] and [15], is given by

$$\widetilde{WR}_i^{\mathrm{D}} = WR_i^{\mathrm{P}}(B_i^{\mathrm{D}} + C_i - (F_i - \Delta)) + (F_i - \Delta), \tag{18}$$

where WR_i^{P} denotes the worst-case response time of τ_i under FPPS. According to [15], Δ is an arbitrary small positive value needed to ensure that the final subjob has actually started. Hence, when task τ_i has consumed $C_i - (F_i - \Delta)$, the final subjob has (just) started.

4.1.2 Existing analysis for FPNS

In this section, we first recapitulate the update of [23] given in [39] to take account of tasks being non-preemptive. Next, we show that the update is very similar to the analysis for FPDS as given by Equation (18).

The non-preemptive nature of tasks may cause blocking of a task by at most one lower priority task. The maximum blocking B_i^{N} of task τ_i by a lower priority task is equal to the longest computation time of a task with a priority lower than task τ_i, i.e.

$$B_i^{\mathrm{N}} = \max_{j>i} C_j. \tag{19}$$

Similarly to B_i^{D}, B_i^{N} is a supremum for all tasks, except for the lowest priority task, i.e. that value cannot be assumed for $i < n$.

The worst-case response time $\widetilde{WR}_i^{\mathrm{N}}$ is given by

$$\widetilde{WR}_i^{\mathrm{N}} = w_i + C_i, \tag{20}$$

where w_i is the smallest $x \in \mathbb{R}^+$ that satisfies

$$x = B_i^{\mathrm{N}} + \sum_{j<i} \left\lceil \frac{x + \tau_{res}}{T_j} \right\rceil C_j. \tag{21}$$

In this latter equation, τ_{res} is the resolution with which time is measured. To calculate w_i, an iterative procedure based on recurrence relationships can be used. An appropriate initial value of this procedure is $w_i^{(0)} = B_i^{\mathrm{N}} + \sum_{j<i} C_j$.

We now show that these results for FPNS are similar to the existing analysis for FPDS. To this end, we substitute $w_i = w'_i - \tau_{res}$, $x = x' - \tau_{res}$, and $\tau_{res} = \Delta$ in equations (20) and (21). Hence, the worst-case response time $\widetilde{WR}_i^{\mathrm{N}}$ is given by

$$\widetilde{WR}_i^{\mathrm{N}} = w'_i + (C_i - \Delta),$$

where w'_i is the smallest $x' \in \mathbb{R}^+$ that satisfies

$$x' = B_i^{\mathrm{N}} + \Delta + \sum_{j<i} \left\lceil \frac{x'}{T_j} \right\rceil C_j.$$

Reusing the results for FPPS, we therefore get

$$\widetilde{WR}_i^{\mathrm{N}} = WR_i^{\mathrm{P}}(B_i^{\mathrm{N}} + \Delta) + (C_i - \Delta). \tag{22}$$

Because we have $F_i = C_i$ and $B_i^{\mathrm{D}} = B_i^{\mathrm{N}}$ for FPNS, Equation (22) for FPNS is similar to Equation (18) for FPDS. There is an aspect requiring further attention, however. In particular, Equation (18) is based on an arbitrary small positive value Δ whereas the analysis for FPNS is based on the resolution τ_{res} with which time is measured. We will return to this issue in Section 8.3.

4.2 Existing analysis is pessimistic

Consider the set $\mathcal{T}_2$ consisting of three tasks with characteristics as described in Table 2. Based on (18) we derive

	T_i	D_i	C_i
τ_1	5	4	2
τ_2	7	7	1 + 2
τ_3	30	30	2 + 2

Table 2. Task characteristics of $\mathcal{T}_2$.

$$\begin{aligned} \widetilde{WR}_2^{\mathrm{D}} &= WR_2^{\mathrm{P}}(B_2^{\mathrm{D}} + C_2 - (F_2 - \Delta)) + (F_2 - \Delta) \\ &= WR_2^{\mathrm{P}}(2 + 3 - (2 - \Delta)) + (2 - \Delta) \\ &= WR_2^{\mathrm{P}}(3 + \Delta) + (2 - \Delta) = 7 + \Delta + (2 - \Delta) = 9. \end{aligned}$$

However, the existing analysis does not take into account that τ_i can only be blocked by a subjob of a lower priority task if that subjob starts *before* the simultaneous release of τ_i and all tasks with a higher priority than τ_i. This aspect can be taken into account in the analysis by replacing B_i^{D} in (18) by $(B_i^{\mathrm{D}} - \Delta)^+$. The notation x^+ is used to indicate that the blocking time can not become negative for the lowest priority task. The worst-case response time of τ_2 now becomes $7 - \Delta$, as illustrated in Figure 4. For $\Delta \downarrow 0$, we therefore find a supremum (and not a maximum) equal to 7 for the worst-case response time of τ_2. As a result, the existing analysis is pessimistic.

4.3 Existing analysis is optimistic

We will give three examples illustrating that the existing analysis is optimistic. For all three examples, deadlines are equal to periods, i.e. $D_i = T_i$. The first section shows an obvious example, i.e. an example with a utilization factor $U > 1$. The second section shows an example with $U < 1$. The third section shows an example with $U = 1$.

For all three examples, the task set consists of just two tasks. For such task sets, the worst-case response time analysis under FPDS presented in [12, 13, 15] and in [11] is very similar. In particular, the worst-case response time $\widetilde{WR}_2^{\mathrm{D}}$ of task τ_2 is determined by looking at the response time of the first job of task τ_2 upon a simultaneous release with task τ_1. However, the worst-case response time of task τ_2 is not assumed for the first job for all three examples.

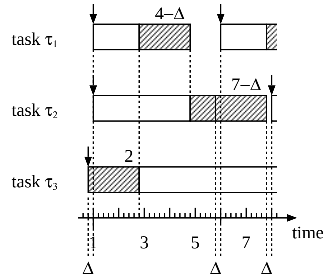

Figure 4. Timeline for $\mathcal{T}_2$ under FPDS with a release of tasks τ_1 and τ_2 at time $t = 1$ and a release of task τ_3 at time $t = 1 - \Delta$.

4.3.1 An example with $U > 1$

An example refuting the worst-case response time analysis is given in Table 3. Note that the utilization factor U of this set of tasks $\mathcal{T}_3$ is given by $U = \frac{2}{5} + \frac{4.5}{7} > 1$. Hence, the task set is not schedulable. Based on (18), we derive

	$T_i = D_i$	C_i
τ_1	5	2
τ_2	7	1.5 + 3

Table 3. Task characteristics of $\mathcal{T}_3$.

$$\begin{aligned} \widetilde{WR}_2^{\mathrm{D}} &= WR_2^{\mathrm{P}}(B_2 + C_2 - (F_2 - \Delta)) + (F_2 - \Delta) \\ &= WR_2^{\mathrm{P}}(0 + 4.5 - (3 - \Delta)) + (3 - \Delta) \\ &= WR_2^{\mathrm{P}}(1.5 + \Delta) + (3 - \Delta) = 3.5 + \Delta + (3 - \Delta) = 6.5. \end{aligned}$$

This value corresponds with the response time of the first job of task τ_2 upon a simultaneous release with task τ_1, as illustrated in Figure 5. However, the same figure also illustrates that the second job of τ_2 misses its deadline. Stated in other words, the existing worst-case response time analysis is optimistic.

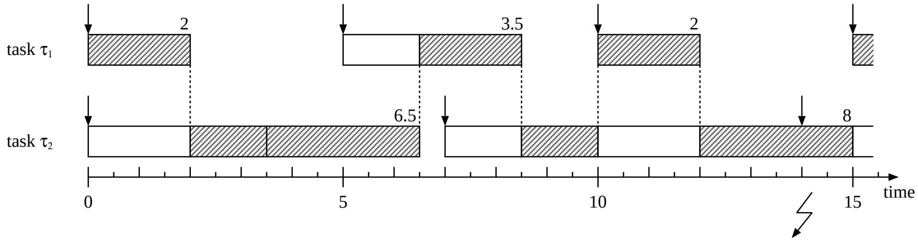

Figure 5. Timeline for $\mathcal{T}_3$ under FPDS with a simultaneous release of both tasks at time zero.

4.3.2 An example with $U < 1$

Another example refuting the worst-case response time analysis is given in Table 4. Note that the utilization factor U of this set of tasks $\mathcal{T}_4$ is given by $U = \frac{2}{5} + \frac{4.1}{7} < 1$. Hence, the task set could be schedulable. Applying (18) yields $\widetilde{WR}_2^{\mathrm{D}} = 6.1$, which corresponds with the response time of the first job of task τ_2 upon a simultaneous release with task τ_1; see Figure 6. However, the same figure also illustrates that the second job of task τ_2 misses its deadline.

4.3.3 An example with $U = 1$

Consider task set $\mathcal{T}_5$ given in Table 5. The utilization factor U of this set of tasks is given by $U = \frac{2}{5} + \frac{4.2}{7} = 1$. The task set is not schedulable by FPPS, as we showed in Section 3 that the task set is only schedulable when C_2 is at most 3. Figure 7 shows a timeline with the executions of these two tasks under FPDS with a simultaneous release at time zero in an interval of length 35, i.e. equal to the *hyperperiod* of the tasks. Applying (18) yields $\widetilde{WR}_2^{\mathrm{D}} = 6.2$, which corresponds with the response time of the first job of task τ_2 in Figure 7. However, the response time of the 5^{th} job of task τ_2 is equal to 7, illustrating once again that the existing analysis is too optimistic. Nevertheless, the task set is schedulable under FPDS for this phasing.

	$T_i = D_i$	C_i
τ_1	5	2
τ_2	7	2 + 2.1

Table 4. Task characteristics of $\mathcal{T}_4$.

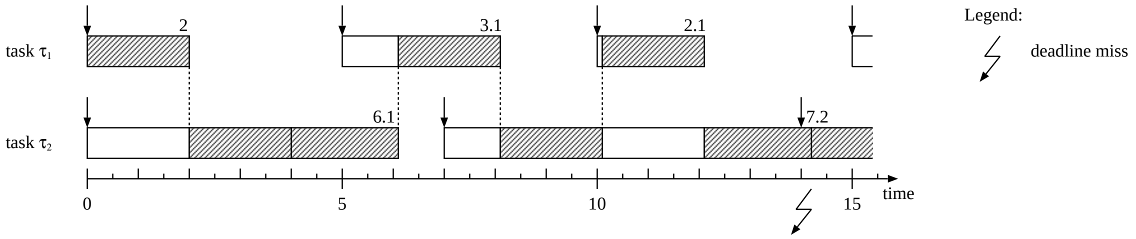

Figure 6. Timeline for $\mathcal{T}_4$ under FPDS with a simultaneous release of all tasks at time zero.

Now, consider task set $\mathcal{T}_6$ given in Table 6, which is similar to task set $\mathcal{T}_5$ given in Table 5, except for the fact that rather than having a second subjob for task τ_2 it has a task τ_3. Figure 8 shows a timeline with the executions of these three tasks under FPNS with a simultaneous release at time zero in an interval of length 35, i.e. equal to the *hyperperiod* of the tasks. Applying (18) yields $\widetilde{WR}_3^{\mathrm{D}} = 6.2$, which corresponds to the response time of the first job of task τ_3 in Figure 8. However, the response time of the 5^{th} job of task τ_3 is equal to 7, illustrating once again that the existing analysis is too optimistic. Nevertheless, the task set is schedulable under FPNS for this phasing.

4.4 Concluding remark

We have shown that we cannot restrict ourselves to the response time of the first job of a task when determining the worst-case response time of that task under FPDS. The reason for this is that the final subjob of a task τ_i can defer the execution of higher priority tasks, which can potentially give rise to higher interference for subsequent jobs of task τ_i. This problem can therefore arise for all tasks, except for the highest priority task. González Harbour et al. [19] identified the same influence of jobs of a task for relative deadlines at most equal to periods in the context of FPPS of periodic tasks with varying execution priority.

Considering Figure 7, we see that every job of task τ_2 in the interval $[0, 26.8)$ defers the execution of a job of task τ_1. Moreover, that deferred job of task τ_1 subsequently gives rise to additional interference for the next job of task τ_2. This situation ends when the job of τ_2 is started at time $t = 28$, i.e. the 5^{th} job of τ_2 does not defer the execution of a job of τ_1. Viewed in a different way, we may state that the active intervals of the jobs of tasks τ_1 and τ_2 overlap in the interval $[0, 35)$. Note that this overlapping starts at time $t = 0$ and ends at time $t = 35$, and we therefore term this interval $[0, 35)$ a *level-2 active period*. Informally, a *level-i active period* is a *smallest* interval that only contains entire active intervals of jobs of task τ_i and jobs of tasks with a higher priority than task τ_i. Hence, the active interval of every job of a task τ_i is contained in a level-*i* active period. To determine the worst-case response time of a task τ_i, we therefore only have to consider level-*i* active periods. However, as illustrated by the examples shown in this section and mentioned above, we cannot restrict ourselves to the response time of the first job of a task τ_i when determining the worst-case response time of that task under FPDS. Instead, we have to consider the response times of *all* jobs in a level-*i* active period. In a subsequent section, we will show that it suffices to consider only the response times of jobs in a level-*i* active period that starts at a so-called ε-critical instant.

5 Active period

This section presents a formal definition of a *level-i active period*, which is based on the notion of *pending load*, and theorems to determine the length of a level-*i* active period. As mentioned before, a level-*i* active period may contain multiple jobs of τ_i. We therefore also define the notion of a *level-(i,k) active period*, and present a theorem to determine the length of such a

	$T_i = D_i$	C_i
τ_1	5	2
τ_2	7	1.2 + 3

Table 5. Task characteristics of $\mathcal{T}_5$.

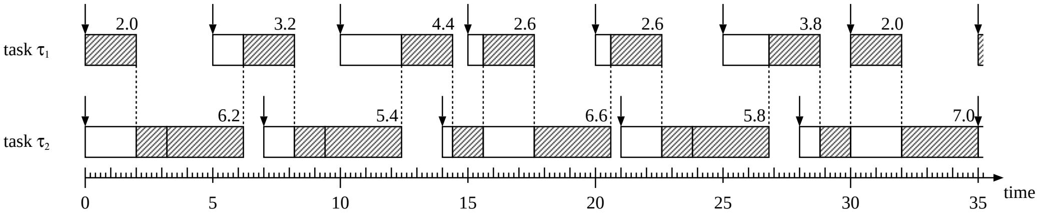

Figure 7. Timeline for $\mathcal{T}_5$ under FPDS with a simultaneous release of all tasks at time zero.

	T_i	C_i
τ_1	5	2
τ_2	7	1.2
τ_3	7	3

Table 6. Task characteristics of $\mathcal{T}_6$.

period. Informally, a level-(i,k) active period is a *smallest* interval that contains k successive active intervals of jobs of task τ_i and all jobs of tasks with a higher priority than task τ_i. These notions and theorems form the basis for the worst-case analysis for FPDS in the next section.

We start with the definition of the notion level-i active period in Section 5.1. Next, we provide examples of level-i active periods in Section 5.2. The length of a level-i active period is the topic of Section 5.3. We refine the notion of level-i active period to level-(i,k) active period in Section 5.4, and conclude with a theorem to determine its length in Section 5.4.3.

5.1 Level-*i* active period

The notion of level-i active period is defined in terms of the notion of *pending load*, which on its turn is defined in terms of the notion of *active job*.

5.1.1 Active job and pending load

Definition 1. *A job k of a task τ_i is* active *at time t if and only if $t \in [a_{ik}, f_{ik})$, where a_{ik} and f_{ik} are the* activation *(or* release*)* time *and the* finalization *(or* completion*)* time *of that job, respectively.* ☐

The *active interval* of job k of task τ_i is defined as the time span between the activation time of that job and its completion, i.e. $[a_{ik}, f_{ik})$. We now define the notion of pending load in terms of active job, and derive properties for the pending load.

Definition 2. *The* pending load $P_i^\tau(t)$ *is the amount of processing at time t that still needs to be performed for the active jobs of tasks τ_i that are released before time t, i.e.*

$$P_i^\tau(t) = \left(\left\lceil \frac{t - \varphi_i}{T_i} \right\rceil \right)^+ \cdot C_i - \int_0^t \sigma_i^\tau(t')dt', \tag{23}$$

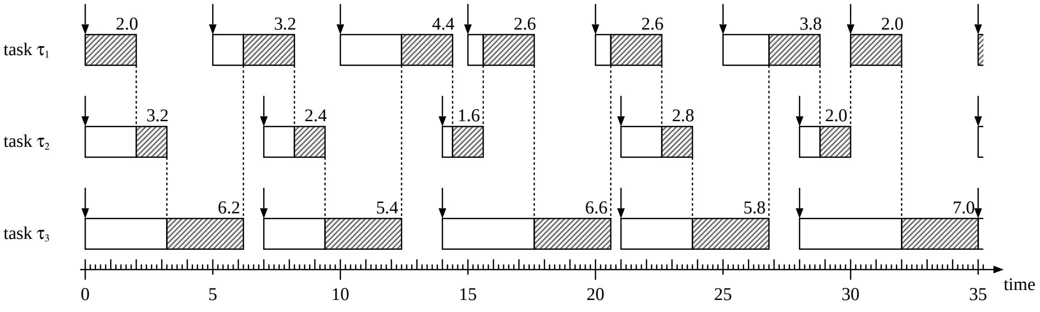

Figure 8. Timeline for $\mathcal{T}_6$ under FPNS with a simultaneous release of all tasks at time zero. The numbers to the top right corner of the boxes denote the response times of the respective releases.

where

$$\sigma_i^\tau(t) = \begin{cases} 1 & \textit{if task } \tau_i \textit{ is being executed at time } t\textit{, i.e. } \sigma(t) = i \\ 0 & \textit{otherwise.} \end{cases}$$

□

Note that the term $\left(\left\lceil \frac{t-\varphi_i}{T_i} \right\rceil\right)^+ \cdot C_i$ in (23) is equal to the amount of processing that needs to be performed due to releases of task τ_i in $[0,t)$. The term $\int_0^t \sigma_i^\tau(t')dt'$ is equal to the amount of processing that has been performed for τ_i. The righthand side of (23) is therefore equal to the amount of processing at time t due to releases of jobs of task τ_i before t that still needs to be performed.

We subsequently define the notions of (*cumulative*) *pending load* $P_i(t)$ and (*processor*) *pending load* $P(t)$.

Definition 3. *The (*cumulative*)* pending load *$P_i(t)$ is the amount of processing at time t that still needs to be performed for the active jobs of tasks τ_j with $j \leq i$ that are released before time t, i.e.*

$$P_i(t) = \sum_{j \leq i} P_j^\tau(t) = \sum_{j \leq i} \left(\left\lceil \frac{t-\varphi_j}{T_j} \right\rceil\right)^+ \cdot C_j - \int_0^t \sigma_i(t')dt', \tag{24}$$

where

$$\sigma_i(t) = \sum_{j \leq i} \sigma_j^\tau(t) = \begin{cases} 1 & \textit{if } \sigma(t) \in \{1, \ldots, i\} \\ 0 & \textit{otherwise.} \end{cases}$$

□

Definition 4. *The (*processor*)* pending load *$P(t)$ is the amount of processing at time t that still needs to be performed for the active jobs of all tasks that are released before time t, i.e.*

$$P(t) = P_n(t). \tag{25}$$

□

Corollary 1. *The* order *in which the tasks τ_j with $j \leq i$ are executed is immaterial for the cumulative pending load P_i.* □

For $i < n$, the cumulative pending load P_i also depends on blocking due to a lower priority task. As an example, let $P_i(t_s) = 0$, then $P_i(t) = C_s$ for all $t \in (t_s, t_s')$ under FPDS if the following three conditions hold:

- a task τ_s with $s \leq i$ is released at time t_s,
- no other releases of τ_j for $j \leq i$ take place in $[t_s, t_s')$, and
- a subjob of a lower priority task is executing at time t_s and blocks task τ_s during $[t_s, t_s')$ due to the non-preemptive nature of the subjob.

Because blocking due to a lower priority task does not play a role for the (processor) pending load, $P(t)$ only depends on the *activations* of tasks.

Corollary 2. *The (processor) pending load $P(t)$ is independent of the scheduling algorithm, provided that the algorithm is non-idling.* □

5.1.2 Definition of a level-*i* active period

We now define the notion of level-*i* active period in terms of the pending load $P_i(t)$.

Definition 5. *A level-i active period is an interval $[t_s, t_e)$ with the following three properties.*

1. $P_i(t_s) = 0$;
2. $P_i(t_e) = 0$;
3. $P_i(t) > 0$ *for all* $t \in (t_s, t_e)$.

□

Let the *blocking time* $B_i(t_s)$ of a level-*i* active period that starts at time t_s be defined as the length of the (optionally empty) initial interval during which the tasks τ_j with $j \leq i$ are blocked by a subjob of a task with a lower priority. Note that $B_n(t_s) = 0$ and $0 \leq B_i(t_s) < B_i^{\mathrm{D}}$ for $i < n$.

Lemma 1. *If a level-i active period starts at time t_s and ends at time t_e, then the following properties hold:*

(i) Tasks τ_j with $j \leq i$ are continuously executing in $[t_s, t_e)$, except for an (optionally empty) initial interval $[t_s, t_s + B_i(t_s))$ during which the tasks are blocked by a lower priority task.

(ii) The length $L_i(t_s)$ of that level-i active period is at least $B_i(t_s) + C_s$, where a task τ_s is released at time t_s.

(iii) The order *in which the tasks τ_j with $j \leq i$ are executed is immaterial for the length $L_i(t_s)$.*

Proof. (i) This property follows immediately from the non-preemptive nature of subjobs and the assumptions for fixed-priority scheduling.

(ii) By definition, $P_i(t_s) = 0$. Because the tasks τ_j with $j \leq i$ are blocked in the (optionally empty) initial interval $[t_s, t_s + B_i(t_s))$, and the level-*i* active period contains at least the active interval of task τ_s, the length $L_i(t_s)$ of that level-*i* active period is at least $B_i(t_s) + C_s$.

(iii) This property follows immediately from the definition of a level-*i* active period and Corollary 1. □

From this definition of the level-*i* active period in terms of the pending load $P_i(t)$, we draw the following conclusion.

Corollary 3. *The level-n active period is independent of the scheduling algorithm, provided that the algorithm is non-idling.* □

Note that a level-*i* active period may, but need not, contain activations of task τ_i. In the sequel, we will call a level-*i* active period that contains an activation of task τ_i a *proper* level-*i* active period. Similarly, we call a level-*i* active period that does not contain an activation of τ_i an *improper* level-*i* active period. Unless explicitly stated otherwise, we use the phrase 'level-*i* active period' to denote a proper level-*i* active period in the remainder of this document.

5.2 Examples

We will now consider two examples, one for FPPS based on the timeline in Figure 3 for $\mathcal{T}_1$ and one for FPDS based on the timeline in Figure 7 for $\mathcal{T}_5$.

Consider Figure 9, with a timeline for $\mathcal{T}_1$ under FPPS, pending loads $P_1(t)$, $P_2^\tau(t)$, and $P_2(t)$, and level-*i* active periods. Note that $P_1(t)$ is equal to $P_1^\tau(t)$ by definition. From the graph for $P_1(t)$, we find that the interval $[0,35)$ contains seven level-1 active periods, corresponding with the seven activations of task τ_1, i.e. $[0,5)$, $[5,7)$, $[10,12)$, $[15,17)$, $[20,22)$, $[25,27)$, and $[30,32)$. The horizontal line fragments in the graph for $P_2^\tau(t)$ are caused by the fact that τ_2 is preempted by a job of task τ_1. From the graph for the pending load $P_2(t)$, we find that the interval $[0,35)$ contains eight level-2 active periods, i.e. $[0,5)$, $[5,7)$, $[7,10)$, $[10,12)$, $[14,19)$, $[20,25)$, $[25,27)$, and $[28,33)$. From these eight level-2 active periods, $[0,5)$, $[7,10)$, $[14,19)$, $[20,25)$, and $[28,33)$ are proper, i.e. contain activations of task τ_2, and $[5,7)$, $[10,12)$, and $[25,27)$ are improper. As mentioned before, the level-2 active period only depends on the activations of τ_1 and τ_2, and is independent of the scheduling algorithm.

Consider Figure 10, with a timeline for $\mathcal{T}_5$ under FPDS, pending loads $P_1(t)$, $P_2^\tau(t)$, and $P_2(t)$, and level-*i* active periods. From the graph for $P_1(t)$, we find that the interval $[0,35)$ contains seven level-1 active periods, corresponding with the seven activations of task τ_1, i.e. $[0,2)$, $[5,8.2)$, $[10,14.4)$, $[15,17.6)$, $[20,22.6)$, $[25,28.8)$, and $[30,32)$. The horizontal line fragments in the graph for $P_1(t)$ are caused by the fact that τ_1 is blocked by a subjob of task τ_2. From the graph for the pending load $P_2(t)$, we find that the interval $[0,35)$ contains a single level-2 active period, i.e. $[0,35)$.

5.3 Length of a level-*i* active period

This section presents three theorems for the length of a level-*i* active period. A first theorem presents a recursive equation for the length of a level-*i* active period. A next theorem states that under the following assumption a level-*i* active period that starts will also end.

Assumption 1. *Either $U < 1$ or $U \leq 1$ and the least common multiple (*lcm*) of the periods of the tasks of $\mathcal{T}$ exists.* □

Hence, the assumption is a sufficient condition to guarantee that a level-*i* active period will end when it starts. Because we assume $\varphi_i \geq 0$ for all $i \leq n$, $P_i(0) = 0$ for all $i \leq n$. We therefore conclude that, when Assumption 1 holds, the timeline consists of a sequence of level-*i* active periods, optionally preceded by and separated by idle-periods. A final theorem provides an iterative procedure to determine the length of a level-*i* active period.

Appendix B shows an example illustrating that the level-*n* active period need not end when Assumption 1 does not hold.

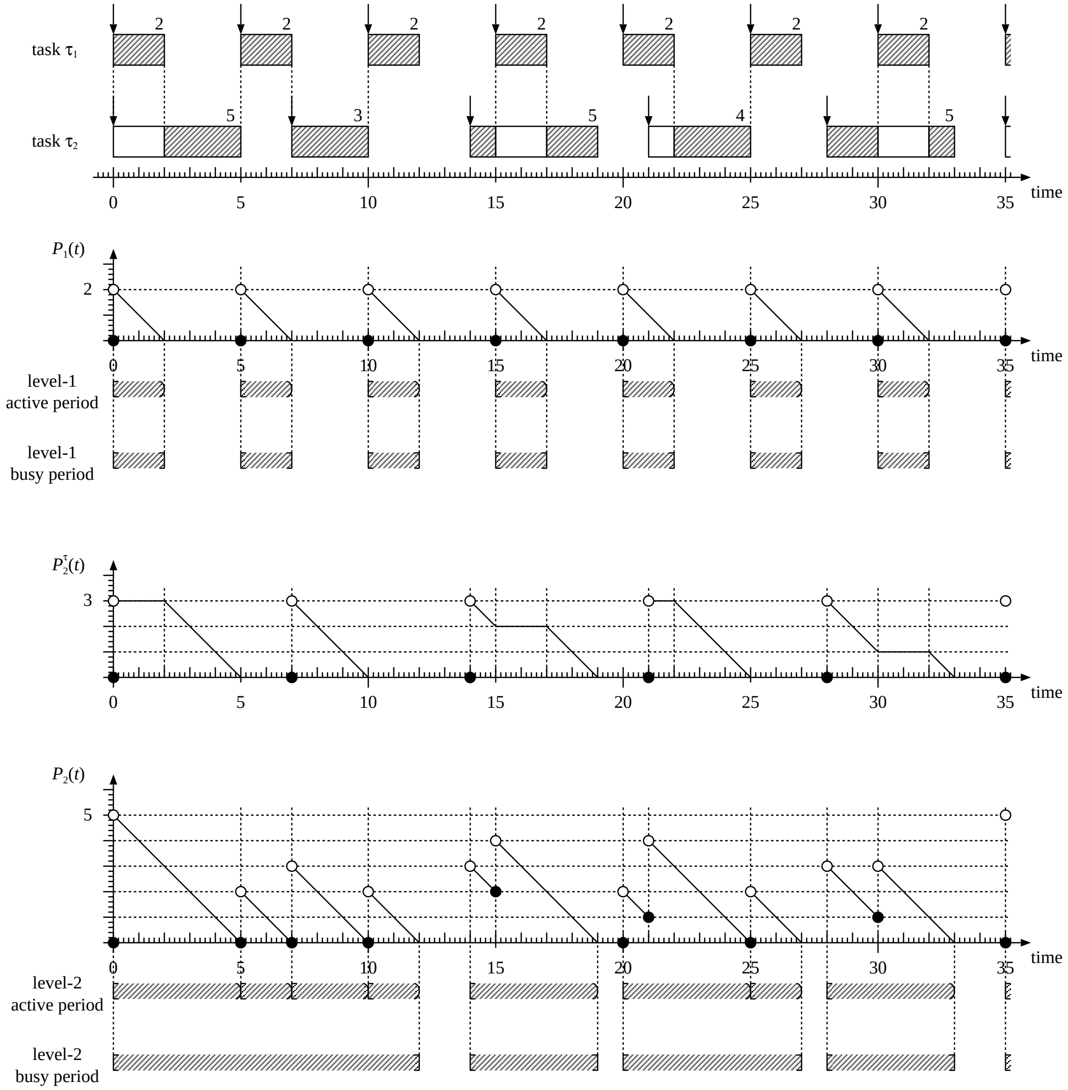

Figure 9. Timeline for $\mathcal{T}_1$ under FPPS, pending loads $P_1(t)$, $P_2^\tau(t)$, and $P_2(t)$, and level-*i* active periods and level-*i* busy periods. From the eight level-2 active periods in the interval $[0,35)$, five are proper, i.e. $[0,5)$, $[7,10)$, $[14,19)$, $[20,25)$, and $[28,33)$ contain activations of task τ_2. The other three are improper, i.e. $[5,7)$, $[10,12)$, and $[25,27)$.

5.3.1 A recursive equation

Theorem 6. *The length $L_i(t_s)$ of a level-i active period that starts at time t_s is found for the smallest $x \in \mathbb{R}^+$ that satisfies the following equation*

$$x = B_i(t_s) + \sum_{j \le i} \left(\left\lceil \frac{x - \varphi_j(t_s)}{T_j} \right\rceil \right)^+ \cdot C_j. \tag{26}$$

Proof. Because the level-*i* active period starts at time t_s, $P_i(t_s) = 0$ by definition. Now assume the level-*i* active period under consideration ends at time t_e. Hence, time t_e is the *smallest* t larger than t_s for which $P_i(t) = 0$, and the length $L_i(t_s)$ of the

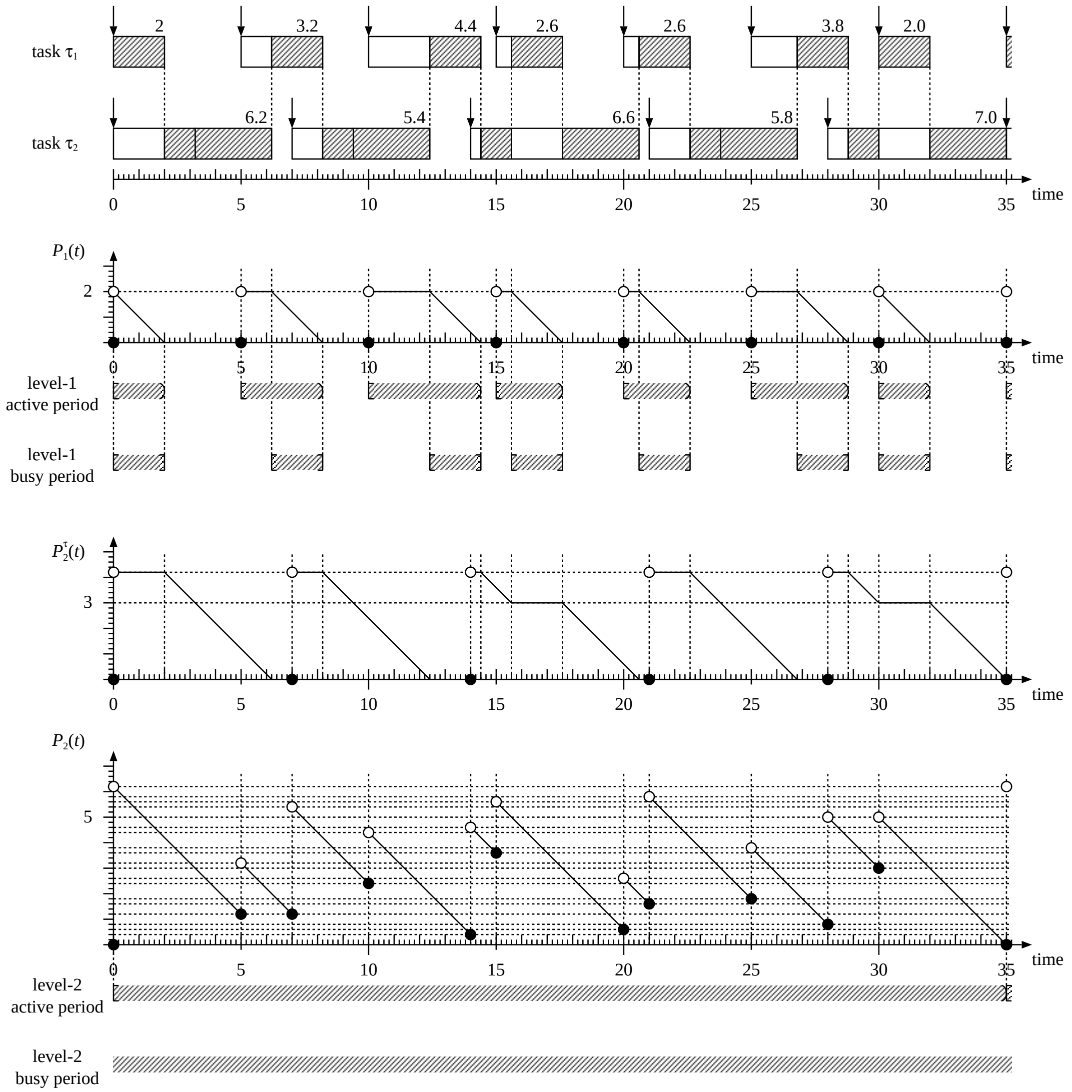

Figure 10. Timeline for $\mathcal{T}_5$ under FPDS, pending loads $P_1(t)$, $P_2^\tau(t)$, and $P_2(t)$, and level-i active periods and level-i busy periods.

active period becomes $t_e - t_s$. We now derive (26) from $P_i(t_e) = 0$.

$$\begin{aligned}
P_i(t_e) &= \{(24)\} \quad \sum_{j\le i}\left(\left\lceil\frac{t_e-\varphi_j}{T_j}\right\rceil\right)^+ \cdot C_j - \int_0^{t_e}\sigma_i(t)dt \\
&= P_i(t_s) + \sum_{j\le i}\left(\left\lceil\frac{t_e-(t_s+\varphi_j(t_s))}{T_j}\right\rceil\right)^+ \cdot C_j - \int_{t_s}^{t_e}\sigma_i(t)dt \\
&= \{P_i(t_s)=0\} \quad \sum_{j\le i}\left(\left\lceil\frac{t_e-(t_s+\varphi_j(t_s))}{T_j}\right\rceil\right)^+ \cdot C_j - \int_{t_s}^{t_e}\sigma_i(t)dt \\
&= 0
\end{aligned}$$

From Lemma 1, we derive that the lower priority task is executing in $[t_s, t_s + B_i(t_s))$, and only tasks τ_j with $j \le i$ are executing in $[t_s + B_i(t_s), t_e)$. Hence, we conclude that

$$\int_{t_s}^{t_e}\sigma_i(t)dt = t_e - (t_s + B_i(t_s)).$$

Substituting this result in the former equation, we get

$$t_e - (t_s + B_i(t_s)) = \sum_{j \leq i} \left(\left\lceil \frac{t_e - (t_s + \varphi_j(t_s))}{T_j} \right\rceil \right)^+ \cdot C_j,$$

and by subsequently substituting $t_e = x + t_s$, we get (26). Because time t_e is the smallest t (larger than t_s) for which $P_i(t) = 0$, $x = t_e - t_s$ is the smallest value in $\mathbb{R}^+$ that satisfies (26), which proves the theorem. □

5.3.2 End of a level-*i* active period

We now present a theorem which states that there exist positive solutions for the recursive equation (26) if Assumption 1 holds. To that end, we will use Lemma 4.3 from [5] (see Lemma 15 in Appendix A), and first prove two lemmas.

Lemma 2. *There exists a positive solution for the recursive equation (26) for the length of the level-i active period if* $U_i < 1$.

Proof. We will prove that the condition $U_i < 1$ is sufficient by means of Lemma 4.3 of [5]. Let f be defined as

$$f(x) = B_i(t_s) + \sum_{j \leq i} \left(\left\lceil \frac{x - \varphi_j(t_s)}{T_j} \right\rceil \right)^+ \cdot C_j.$$

We choose $a = \min\limits_{l \leq i} \frac{C_l}{2}$, hence

$$f(a) = f(\min_{l \leq i} \frac{C_l}{2}) = B_i(t_s) + \sum_{j \leq i} \left(\left\lceil \frac{\min_{l \leq i} \frac{C_l}{2} - \varphi_j(t_s)}{T_j} \right\rceil \right)^+ \cdot C_j.$$

By definition, there is at least one task that is released at the start of the level-i active period. Let task τ_k with $k \leq i$ be released at time t_s, i.e. $\varphi_k(t_s) = 0$. We now get

$$f(a) \geq B_i(t_s) + \left\lceil \frac{\min_{l \leq i} \frac{C_l}{2}}{T_k} \right\rceil C_k = B_i(t_s) + C_k > \min_{l \leq i} \frac{C_l}{2} = a,$$

hence $f(a) > a$. In order to choose an appropriate b, we make the following derivation.

$$f(x) \leq B_i(t_s) + \sum_{j \leq i} \left\lceil \frac{x}{T_j} \right\rceil C_j < B_i(t_s) + \sum_{j \leq i} (\frac{x}{T_j} + 1) C_j = B_i(t_s) + xU_i + \sum_{j \leq i} C_j.$$

As $U_i < 1$, the relation

$$x \geq B_i(t_s) + xU_i + \sum_{j \leq i} C_j$$

holds for

$$x \geq \frac{B_i(t_s) + \sum\limits_{j \leq i} C_j}{1 - U_i}.$$

We now choose

$$b = \frac{B_i(t_s) + \sum\limits_{j \leq i} C_j}{1 - U_i},$$

and therefore get $b > f(b)$. Now the conditions for Lemma 15 hold, i.e. the function $f(x)$ is defined and strictly non-decreasing in an interval $[a, b]$ with $f(a) > a$ and $f(b) < b$. Hence, there exists an

$$x \in (\min_{l \leq i} \frac{C_l}{2}, \frac{B_i(t_s) + \sum\limits_{j \leq i} C_j}{(1 - U_i)})$$

such that $x = f(x)$. □

Lemma 3. *There exists a positive solution for the recursive equation (26) for the length of the level-n active period if* $U = 1$ *and the least common multiple of the periods of* $\mathcal{T}$ *exists.*

Proof. We first observe that $B_n(t_s) = 0$ for the level-*n* active period, i.e. the lowest priority task is never blocked. Next, we distinguish two complementary cases, a first case with $\varphi_i(t_s) = 0$ for all i and a second case where this does not hold. We prove the lemma by considering both cases separately.

For the first case, we prove that for $B_i(t_s) = 0$ and $\varphi_i(t_s) = 0$ for all i the value $x = \mathrm{lcm}(T_1, \dots, T_n)$ is a solution of (26). For these values of $B_i(t_s)$ and $\varphi_i(t_s)$, equation (26) simplifies to

$$x = \sum_{j \le n} \left\lceil \frac{x}{T_j} \right\rceil C_j .$$

Because $\left\lceil \frac{\mathrm{lcm}(T_1,\dots,T_n)}{T_j} \right\rceil C_j = \mathrm{lcm}(T_1, \dots, T_n) \frac{C_j}{T_j}$ and $\sum\limits_{j \le n} \frac{C_j}{T_j} = U = 1$, we immediately see that $\mathrm{lcm}(T_1, \dots, T_n)$ is a (positive) solution.

For the second case, we prove that the condition $U = 1$ and lcm of the periods of $\mathcal{T}$ exists is sufficient by means of Lemma 15. Let f be defined as

$$f(x) = \sum_{j \le n} \left(\left\lceil \frac{x - \varphi_j(t_s)}{T_j} \right\rceil \right)^+ \cdot C_j .$$

We choose $a = \min_{j \le n} C_j/2$. Similar to the proof of Lemma 2, we find $f(a) > a$. In order to choose an appropriate b, we make the following derivation.

$$f(x) \le \sum_{j \le n} \left\lceil \frac{x}{T_j} \right\rceil C_j$$

We now consider two disjunct cases for $x = \mathrm{lcm}(T_1, \dots, T_n)$. If $f(x) < \sum_{j \le n} \left\lceil \frac{x}{T_j} \right\rceil C_j$, we choose $b = \mathrm{lcm}(T_1, \dots, T_n)$, and therefore get $b > f(b)$. Now the conditions for Lemma 15 hold, i.e. the function $f(x)$ is defined and strictly non-decreasing in an interval $[a, b]$ with $f(a) > a$ and $f(b) < b$. Hence, there exists an $x \in (\min_{j \le n} \frac{C_j}{2}, \mathrm{lcm}(T_1, \dots, T_n))$ such that $x = f(x)$. If $f(x) = \sum_{j \le i} \left\lceil \frac{x}{T_j} \right\rceil C_j$, we found a (positive) solution and we are also done. ☐

Appendix B.1 presents an example consisting of two tasks with $U = 1$ and the least common multiple of the periods does not exist, where the level-*n* active period does not end.

Theorem 7. *If Assumption 1 holds, a level-i active period that is started at time* t_s *is guaranteed to end.*

Proof. The theorem follows immediately from Lemmas 2 and 3. ☐

5.3.3 An iterative procedure

The next theorem provides an iterative procedure to determine the length of a level-*i* active period.

Theorem 8. *Let the level-i active period start with a release of a task* τ_s *at time* t_s*. If Assumption 1 holds, the length* $L_i(t_s)$ *of that level-i active period can be found by the following iterative procedure.*

$$L_i^{(0)}(t_s) = B_i(t_s) + C_s \tag{27}$$

$$L_i^{(l+1)}(t_s) = B_i(t_s) + \sum_{j \le i} \left(\left\lceil \frac{L_i^{(l)}(t_s) - \varphi_j(t_s)}{T_j} \right\rceil \right)^+ \cdot C_j, \quad l = 0, 1, \dots \tag{28}$$

Proof. From Lemma 2 and Lemma 3, we know that there exists a positive solution of Equation (26) when Assumption 1 holds. To prove the theorem, we first prove that the sequence is non-decreasing. Next, we prove that the procedure stops when the length $L_i(t_s)$ is reached, i.e. for the smallest solution of Equation (26). To that end, we show that all values in the sequence $L_i^{(l)}(t_s)$ are lower bounds on $L_i(t_s)$. To show that the procedure terminates, we show that the sequence can only take a finite number of values to reach that solution.

We prove that the sequence is non-decreasing, by induction. To this end,we start by noting that $L_i^{(0)}(t_s) = B_i(t_s) + C_s > 0$, and

$$\begin{aligned} L_i^{(1)}(t_s) &= B_i(t_s) + \sum_{j \le i} \left(\left\lceil \frac{L_i^{(0)}(t_s) - \varphi_j(t_s)}{T_j} \right\rceil \right)^+ \cdot C_j \\ &\ge \{\varphi_s(t_s) = 0\}\ B_i(t_s) + C_s = L_i^{(0)}(t_s). \end{aligned}$$

Next, if $L_i^{(l+1)}(t_s) \ge L_i^{(l)}(t_s)$, then we can conclude from Equation (28) that also $L_i^{(l+2)}(t_s) \ge L_i^{(l+1)}(t_s)$, as filling in a higher value in the right-hand side of Equation (28) gives a higher or equal result.

We next prove $L_i^{(l)}(t_s) \le L_i(t_s)$, for all $l = 0, 1, \ldots$, by induction. From Lemma 1 item (ii) we know $L_i^{(0)}(t_s) = B_i(t_s) + C_s \le L_i(t_s)$. Next, if $L_i^{(l)}(t_s)$ is a lower bound on $L_i(t_s)$, then

$$\sum_{j \le i} \left(\left\lceil \frac{L_i^{(l)}(t_s) - \varphi_j(t_s)}{T_j} \right\rceil \right)^+ \cdot C_j$$

is a lower bound on the amount of processing that needs to be performed due to releases of task τ_i and its higher priority tasks in the interval of length $L_i^{(l)}(t_s)$, and hence $L_i^{(l+1)}(t_s)$ is also a lower bound on $L_i(t_s)$.

Finally, we prove that the sequence can only take on a finite number of values. To this end, we note that $L_i^{(l)}(t_s)$ is bounded from below by $B_i(t_s) + C_s$ and from above by the solution. □

5.4 Level-(i,k) active period

Similar to a level-i active period, a level-(i,k) active period is defined in terms of the notion pending load. For the definition of a level-(i,k) active period, we first need to refine the notion of pending load. We assume in this section that Assumption 1 holds.

5.4.1 A refinement of pending load

Let a level-i active period start at time t_s. As described above, the length of that period is given by the smallest $x > 0$ satisfying (26). Let the length of that period be $L_i(t_s)$. The number of jobs $l_i(t_s)$ of task τ_i in that period is now given by

$$l_i(t_s) = \left\lceil \frac{L_i(t_s) - \varphi_i(t_s)}{T_i} \right\rceil. \tag{29}$$

We now refine our notion of pending load $P_i(t)$ by considering only the first $k+1 \le l_i(t_s)$ jobs of τ_i in the active period, where $k \in \mathbb{N}$.

Definition 6. *The* pending load $P_{ik}(t)$ *in a level-i active period that started at time* $t_s < t$ *and ends at time* $t_e \ge t$ *is the amount of processing at time t that still needs to be performed for at most the first* $k+1 \le l_i(t_s)$ *jobs of* τ_i *and the jobs of tasks* τ_j *with* $j < i$ *that are released in* $[t_s, t)$*, i.e.*

$$P_{ik}(t) = \min(\left(\left\lceil \frac{t - (t_s + \varphi_i(t_s))}{T_i} \right\rceil \right)^+, k+1) \cdot C_i + \sum_{j<i} \left(\left\lceil \frac{t - (t_s + \varphi_j(t_s))}{T_j} \right\rceil \right)^+ \cdot C_j - \int_{t_s}^{t} \sigma_i(t')dt', \tag{30}$$

with $\sigma_i(t)$ *as defined in Definition 3. At the start of a level-i active period and outside level-i active periods,* $P_{ik}(t)$ *is equal to zero.* □

5.4.2 Definition of a level-(i,k) active period

Similarly, we refine our notion of level-i active period to *level-*(i,k) *active period.*

Definition 7. *A level-*(i,k) *active period is an interval* $[t_s, t_e)$ *with the following three properties.*

1. $P_{ik}(t_s) = 0$;
2. $P_{ik}(t_e) = 0$;
3. $P_{ik}(t) > 0$ *for* $t \in (t_s, t_e)$.

□

5.4.3 Length of a level-(i,k) active period

Theorem 9. *Let the number of jobs of task τ_i in a level-i active period that starts at time t_s be given by $l_i(t_s)$. The length $L_{ik}(t_s)$ of that level-(i,k) active period with $0 \leq k < l_i(t_s)$ is the smallest $x \in \mathbb{R}^+$ satisfies the following equation*

$$x = B_i(t_s) + (k+1)C_i + \sum_{j<i} \left(\left\lceil \frac{x - \varphi_j(t_s)}{T_j} \right\rceil \right)^+ \cdot C_j. \tag{31}$$

Proof. The proof is similar to the proof of Theorem 6. □

6 Worst-case analysis for FPDS

This section provides theorems for the notions of critical instant and worst-case response times for tasks under FPDS and arbitrary phasing, and theorems to determine the worst-case response times analytically. We assume in this section that Assumption 1 holds. Moreover, we consider an arbitrary level-i active period with a start at time t_s. As described in Section 2.3, we will use abbreviated representations for the relative notions using a prime ($'$) to denote the value of these notions relative to time t_s, e.g. we use a'_{ik} to denote $a_{ik}(t_s)$.

6.1 A critical instant

Similar to Equation (1), the worst-case response time WR_i^{D} of a task τ_i under FPDS is the largest response time under arbitrary phasing, i.e.

$$WR_i^{\mathrm{D}} = \sup_{\varphi,k} r_{ik}.$$

We can refine this equation by taking blocking of tasks and the notion of level-i active period into account. In particular, we observe that all active intervals of jobs of task τ_i are contained in level-i active periods. Assuming the start of an arbitrary level-i active period at time t_s, the worst-case response time WR_i^{D} of task τ_i can therefore be described as

$$WR_i^{\mathrm{D}} = \sup_{B'_i,\varphi'_1,\ldots,\varphi'_i} \max_{0 \leq k < l'_i(B'_i,\varphi'_1,\ldots,\varphi'_i)} r'_{ik}(B'_i,\varphi'_1,\ldots,\varphi'_i), \tag{32}$$

where l'_i is the number of jobs of task τ_i in that level-i active period.

We will now first present a lemma to determine the response time of job k of task τ_i in a level-i active period. We subsequently present a theorem which states that given an infinitesimal time $\varepsilon > 0$, the maximum response time of task τ_i is assumed in a level-i active period which starts at an ε-*critical instant*. A next theorem refines Equation (32).

Lemma 4. *The response time r'_{ik} of job k of task τ_i in a level-i active period that starts at time t_s with $0 \leq k < l'_i$ and l'_i the number of jobs of task τ_i in that level-i active period is given by*

$$r'_{ik}(B'_i,\varphi'_1,\ldots,\varphi'_i) = b'_{ik,m_i}(B'_i,\varphi'_1,\ldots,\varphi'_{i-1}) + F_i - (kT_i + \varphi'_i), \tag{33}$$

where b'_{ik,m_i} is the relative begin time of the final subjob of job k, given by the smallest non-negative $x \in \mathbb{R}$ satisfying

$$x = B'_i + (k+1)C_i - F_i + \sum_{j<i} \left(\left\lfloor \frac{x - \varphi'_j}{T_j} \right\rfloor + 1 \right)^+ \cdot C_j. \tag{34}$$

Proof. We first look at the relative begin time b'_{ik,m_i} of the final subjob of that job k, and subsequently describe r'_{ik} in terms of the relative begin time, the relative activation time a'_{ik} and the computation time F_i of that final subjob.

The final subjob of job k of task τ_i in the level-i active period that starts at time t_s can begin at time $t_s + b'_{ik,m_i}$ when

- the blocking subjob of the lower priority task has executed B'_i;
- all higher priority tasks that are released in $[t_s, t_s + b'_{ik,m_i}]$ have a completion in that interval;
- all earlier jobs of task τ_i and all earlier subjobs of job k that are released in $[t_s, t_s + b'_{ik,m_i}]$ have a completion in that interval.

Note that the *order* in which the subjobs in the interval $[t_s, t_s + b'_{ik,m_i}]$ are executed is irrelevant for the begin time of the final subjob of job k of task τ_i. Stated in other words, the final subjob of job k of task τ_i can start for the smallest $t_s + x \geq t_s + \max(B'_i, a'_{ik})$ for which $\lim_{t \downarrow t_s + x} P_{ik}(t) = F_i$. We now derive

$$\begin{aligned}
\lim_{t \downarrow t_s + x} P_{ik}(t) &= \{(30)\} \lim_{t \downarrow t_s + x} \left(\min(\left(\left\lceil \frac{t - (t_s + \varphi'_i)}{T_i} \right\rceil \right)^+, k+1) \cdot C_i + \sum_{j<i} \left(\left\lceil \frac{t - (t_s + \varphi'_j)}{T_j} \right\rceil \right)^+ \cdot C_j - \int_{t_s}^{t} \sigma_i(t')dt' \right) \\
&= \min(\left(\lim_{t \downarrow t_s + x} \left\lceil \frac{t - (t_s + \varphi'_i)}{T_i} \right\rceil \right)^+, k+1) \cdot C_i + \sum_{j<i} \left(\lim_{t \downarrow t_s + x} \left\lceil \frac{t - (t_s + \varphi'_j)}{T_j} \right\rceil \right)^+ \cdot C_j - \int_{t_s}^{t_s + x} \sigma_i(t')dt' \\
&= \{\text{Lemma 16}\} \ \min(\left(\left\lfloor \frac{x - \varphi'_i}{T_i} \right\rfloor + 1 \right)^+, k+1) \cdot C_i + \sum_{j<i} \left(\left\lfloor \frac{x - \varphi'_j}{T_j} \right\rfloor + 1 \right)^+ \cdot C_j - \int_{t_s}^{t_s + x} \sigma_i(t')dt' \\
&= \{x \geq \max(B'_i, \varphi'_i + k \cdot T_i)\} \ (k+1) \cdot C_i + \sum_{j<i} \left(\left\lfloor \frac{x - \varphi'_j}{T_j} \right\rfloor + 1 \right)^+ \cdot C_j - (x - B'_i) \\
&= F_i.
\end{aligned}$$

The relative begin time $b'_{ik,m_i}(B'_i, \varphi'_1, \ldots, \varphi'_{i-1})$ is therefore the smallest non-negative $x \in \mathbb{R}$ satisfying the following equation:

$$x = B'_i + (k+1)C_i - F_i + \sum_{j<i} \left(\left\lfloor \frac{x - \varphi'_j}{T_j} \right\rfloor + 1 \right)^+ \cdot C_j.$$

The relative completion time f'_{ik} of job k of τ_i is now given by the relative begin time b'_{ik,m_i} plus the computation time F_i, i.e. $f'_{ik} = b'_{ik,m_i} + F_i$. The response time r'_{ik} of the job k is given by the relative completion time f'_{ik} minus the relative activation time a'_{ik}, i.e.

$$r'_{ik}(B'_i, \varphi'_1, \ldots, \varphi'_i) = b'_{ik,m_i}(B'_i, \varphi'_1, \ldots, \varphi'_{i-1}) + F_i - (kT_i + \varphi'_i).$$

□

Theorem 10. *Given an infinitesimal time $\varepsilon > 0$, the maximum response time of task τ_i under FPDS and arbitrary phasing is assumed when the level-i active period is started at an* ε-critical instant, *i.e. when τ_i has a simultaneous release with all higher priority tasks and a subjob of the lower priority tasks with computation time B_i^{D} starts a time ε before that simultaneous release.*

Proof. Let $R'_i(B'_i, \varphi'_1, \ldots, \varphi'_i)$ denote $\max_{0 \leq k < l'_i(B'_i, \varphi'_1, \ldots, \varphi'_i)} r'_{ik}(B'_i, \varphi'_1, \ldots, \varphi'_i)$. We will prove that $R'_i(B'_i, \varphi'_1, \ldots, \varphi'_i)$ assumes a maximum for $\varphi'_j = 0$ with $j \leq i$ and $B'_i = \left(B_i^{\mathrm{D}} - \varepsilon\right)^+$. Hence, the maximum is assumed when τ_i has a simultaneous release with all higher priority tasks, and a subjob of a lower priority task with computation time B_i^{D} starts an infinitesimal time $\varepsilon > 0$ before that simultaneous release, which proves the theorem.

Based on Theorem 7, i.e. termination of a level-i active period under Assumption 1, we conclude that

- only a finite number of jobs need to be considered to determine the worst-case response time of task τ_i;
- every job of τ_i in a level-i active period has a finite response time.

We will now look at the value of the length L'_i of the level-i active period, the number l'_i of jobs of task τ_i in the level-i active period, and the response time r'_{ik} as a function of the relative phasing φ'_j with $j \leq i$ and the blocking time B'_i. Consider Equation (26) for the length L'_i of a level-i active period. The term $\left\lceil \frac{x - \varphi'_j}{T_j} \right\rceil$ in that equation is a strictly non-increasing function of φ'_j with $j \leq i$. Because $\varphi'_j \geq 0$, a maximum of that term is assumed for $\varphi'_j = 0$. Moreover, the righthand side of Equation (26) is a strictly increasing function of B'_i, and the length L'_i is therefore also a strictly increasing function of B'_i. The largest value of L'_i is found for the largest value of B'_i under consideration, i.e. for $B'_i = \left(B_i^{\mathrm{D}} - \varepsilon\right)^+$. As a consequence, L'_i assumes a maximum for $\varphi'_j = 0$ for all $j \leq i$ and $B'_i = \left(B_i^{\mathrm{D}} - \varepsilon\right)^+$.

Given the behavior of L'_i and Equation (29), we conclude that the number of jobs l'_i of task τ_i in the level-i active period is a strictly non-increasing function of φ'_j with $j \leq i$ and a strictly non-decreasing function of B'_i. As a consequence, l'_i assumes a maximum for $\varphi'_j = 0$ for all $j \leq i$ and $B'_i = \left(B_i^{\mathrm{D}} - \varepsilon\right)^+$.

From Equation (33), we conclude that $r'_{ik}(B'_i,\varphi'_1,\ldots,\varphi'_i)$ is a strictly decreasing function of φ'_i. Because $\varphi'_i \geq 0$, a maximum is assumed for $\varphi'_i = 0$. Now consider Equation (34) for the relative begin time b'_{ik,m_i}. The term $\left\lfloor \frac{x-\varphi'_j}{T_j} \right\rfloor$ in that equation is a strictly non-increasing function of φ'_j. Similarly to φ'_i, $\varphi'_j \geq 0$, a maximum of that term is therefore assumed for $\varphi'_j = 0$. Hence, $b'_{ik,m_i}(B'_i,0,\ldots,0)$ dominates $b'_{ik,m_i}(B'_i,\varphi'_1,\ldots,\varphi'_{i-1})$ for all values of B'_i and all values of φ'_j with $j < i$. Moreover, the righthand side of Equation (34) is a strictly increasing function of B'_i, and $b'_{ik,m_i}(B'_i,0,\ldots,0)$ is therefore also a strictly increasing function of B'_i. The largest value of $b'_{ik,m_i}(B'_i,0,\ldots,0)$ is found for the largest value of B'_i under consideration, i.e. for $B'_i = \left(B^{\mathrm{D}}_i - \varepsilon\right)^+$. As a consequence, $r'_{ik}(B'_i,\varphi'_1,\ldots,\varphi'_i)$ also assumes a maximum for $\varphi'_j = 0$ for all $j \leq i$ and $B'_i = \left(B^{\mathrm{D}}_i - \varepsilon\right)^+$.

From the values of L'_i, l'_i and r'_{ik} as a function of the relative phasing φ'_j with $j \leq i$ and the blocking time B'_i, we conclude that $R'_i(B'_i,\varphi'_1,\ldots,\varphi'_i)$ is a strictly non-increasing function of φ'_1, …, φ'_{i-1}, a strictly decreasing function of φ'_i, and a strictly increasing function of B'_i. As a result, $R'_i(B'_i,\varphi'_1,\ldots,\varphi'_i)$ assumes a maximum for $\varphi'_j = 0$ with $j \leq i$ and $B'_i = \left(B^{\mathrm{D}}_i - \varepsilon\right)^+$, which proves the theorem. □

Theorem 11. *The worst-case response time WR^{D}_i of task τ_i under FPDS and arbitrary phasing is given by*

$$WR^{\mathrm{D}}_i = \lim_{\varepsilon \downarrow 0} \max_{0 \leq k < l'_i((B^{\mathrm{D}}_i-\varepsilon)^+,0,\ldots,0)} r'_{ik}\left(\left(B^{\mathrm{D}}_i - \varepsilon\right)^+, 0, \ldots, 0\right). \tag{35}$$

Proof. Once again, let $R'_i(B'_i,\varphi'_1,\ldots,\varphi'_i)$ denote $\max_{0 \leq k < l'_i((B^{\mathrm{D}}_i-\varepsilon)^+,0,\ldots,0)} r'_{ik}\left(\left(B^{\mathrm{D}}_i - \varepsilon\right)^+, 0, \ldots, 0\right)$. From the proof of Theorem 10, we derive that $R'_i(B'_i,0,\ldots,0)$ dominates $R'_i(B'_i,\varphi'_1,\ldots,\varphi'_i)$ for all values of B'_i and all values of φ'_j with $j \leq i$, i.e.

$$\begin{aligned} WR^{\mathrm{D}}_i &= \sup_{B'_i,\varphi'_1,\ldots,\varphi'_i} R'_i(B'_i,\varphi'_1,\ldots,\varphi'_i) \\ &= \sup_{B'_i} R'_i(B'_i,0,\ldots,0) \end{aligned}$$

Moreover, $R'_i(B'_i,\varphi'_1,\ldots,\varphi'_i)$ is a strictly increasing, i.e. monotonic, function of B'_i. Hence,

$$\begin{aligned} WR^{\mathrm{D}}_i &= \sup_{B'_i} R'_i(B'_i,0,\ldots,0) \\ &= \lim_{\varepsilon \downarrow 0} R'_i\left(\left(B^{\mathrm{D}}_i - \varepsilon\right)^+, 0, \ldots, 0\right), \end{aligned}$$

which proves the theorem. □

In the sequel, we will omit trailing zeros in the parameter list, e.g. we write $r'_{ik}\left(\left(B^{\mathrm{D}}_i - \varepsilon\right)^+\right)$ when $\varphi'_j = 0$ for $j \leq i$.

From the previous two theorems, we draw the following conclusions.

Corollary 4. *The worst-case response time WR^{D}_i is a supremum (and not a maximum) for all tasks, except for the lowest priority task, i.e. that value cannot be assumed for $i < n$.* □

Corollary 5. *A critical instant is a supremum for all tasks, except for the lowest priority task, i.e. that instant cannot be assumed for $i < n$.* □

6.2 Worst-case response times

The next theorem describes WR^{D}_i in terms of the worst-case response time WR^{P}_i and worst-case occupied time WO^{P}_i under FPPS.

First, we present definitions and prove three lemmas for for the worst-case length WL^{D}_i of a level-i active period, the maximum number wl^{D}_i jobs of task τ_i in a level-i active period, and the worst-case response time WR^{D}_{ik} of job k of task τ_i.

Definition 8. *The worst-case length WL^{D}_i of level-i active period under FPDS is the largest length of that period under arbitrary phasing, i.e.*

$$WL^{\mathrm{D}}_i = \sup_{B'_i,\varphi'_1,\ldots,\varphi'_i} L'_i(B'_i,\varphi'_1,\ldots,\varphi'_i). \tag{36}$$

□

Definition 9. *The worst-case number wl_i^{D} of jobs of task τ_i in a level-i active period under FPDS is the largest number in that period under arbitrary phasing, i.e.*

$$wl_i^{\mathrm{D}} = \sup_{B'_i,\varphi'_1,\ldots,\varphi'_i} l'_i(B'_i,\varphi'_1,\ldots,\varphi'_i). \tag{37}$$

□

Definition 10. *The worst-case response time WR_{ik}^{D} of job k of task τ_i, with $1 \le k < wl_i^{\mathrm{D}}$, under FPDS is the largest response time of job k of τ_i under arbitrary phasing, i.e.*

$$WR_{ik}^{\mathrm{D}} = \sup_{B'_i,\varphi'_1,\ldots,\varphi'_i} r'_{ik}(B'_i,\varphi'_1,\ldots,\varphi'_i). \tag{38}$$

□

Lemma 5. *The worst-case length WL_i^{D} of a level-i active period with $i \le n$ under FPDS is given by the smallest $x \in \mathbb{R}^+$ that satisfies the following equation*

$$x = B_i^{\mathrm{D}} + \sum_{j \le i} \left\lceil \frac{x}{T_j} \right\rceil C_j. \tag{39}$$

Proof. The term $\left\lceil \frac{x-\varphi'_j}{T_j} \right\rceil$ in Equation (26) is a strictly non-increasing function of φ'_j with $j \le i$. Because $\varphi'_j \ge 0$, a maximum of that term is assumed for $\varphi'_j = 0$. Now let $L'_i(B'_i)$ denote the length of a level-i active period with $i \le n$ for a simultaneous release of task τ_i with all tasks with a higher priority. Hence, $L'_i(B'_i)$ is the smallest $x \in \mathbb{R}^+$ satisfying equation (26) with $\varphi'_j = 0$, i.e. the smallest $x \in \mathbb{R}^+$ satisfying

$$x = B'_i + \sum_{j \le i} \left\lceil \frac{x}{T_j} \right\rceil C_j. \tag{40}$$

We will now consider the cases $i < n$ and $i = n$ separately.

$\{i = n\}$ The lowest priority task is never blocked, therefore $B_n^{\mathrm{D}} = 0$, and we immediately get (39) by substituting $B'_i = 0$ in equation (40) for $i = n$.

$\{i < n\}$ The righthand side of equation (40) is a strictly increasing function of B'_i, and $L'_i(B'_i)$ is therefore also a strictly increasing function of B'_i. The largest value for $L'_i(B'_i)$ is found for the largest value of $B'_i < B_i^{\mathrm{D}}$. Hence, WL_i^{D} is given by

$$WL_i^{\mathrm{D}} = \lim_{B'_i \uparrow B_i^{\mathrm{D}}} L'_i(B'_i). \tag{41}$$

Given Lemma 17, we can make the following derivation starting from this equation.

$$\begin{aligned}
WL_i^{\mathrm{D}} &= \{(40)\} \lim_{B'_i \uparrow B_i^{\mathrm{D}}} \left(B'_i + \sum_{j \le i} \left\lceil \frac{L'_i(B'_i)}{T_j} \right\rceil C_j \right) \\
&= B_i^{\mathrm{D}} + \sum_{j \le i} \lim_{B'_i \uparrow B_i^{\mathrm{D}}} \left\lceil \frac{L'_i(B'_i)}{T_j} \right\rceil C_j \\
&= \{\text{Lemma 17}\}\ B_i^{\mathrm{D}} + \sum_{j \le i} \left\lceil \lim_{B'_i \uparrow B_i^{\mathrm{D}}} \frac{L'_i(B'_i)}{T_j} \right\rceil C_j \\
&= \{(41)\}\ B_i^{\mathrm{D}} + \sum_{j \le i} \left\lceil \frac{WL_i^{\mathrm{D}}}{T_j} \right\rceil C_j
\end{aligned}$$

Hence, the worst-case length WL_i^{D} is the smallest $x \in \mathbb{R}^+$ satisfying (39), which proves the lemma. □

Because B_i^{D} is a supremum (and not a maximum) for all tasks, except for the lowest priority task, we draw the following conclusion from the previous lemma.

Corollary 6. *The worst-case length WL_i^{D} is a supremum (and not a maximum) for all tasks, except for the lowest priority task, i.e. that value cannot be assumed for $i < n$.* □

Lemma 6. *The maximum number wl_i^{D} of jobs of task τ_i in a level-i active period with $i \leq n$ under FPDS is given by*

$$wl_i^{\mathrm{D}} = \left\lceil \frac{WL_i^{\mathrm{D}}}{T_i} \right\rceil. \tag{42}$$

Proof. We first derive Equation (42) and subsequently prove that wl_i^{D} is a maximum.

As described in the proof of Theorem 10, $l'_i(B'_i)$ is a strictly non-decreasing function of the blocking time B'_i. Because B_i^{D} is a supremum that cannot be assumed, the largest value for $l'_i(B'_i)$ is therefore found for the largest value of $B'_i < B_i^{\mathrm{D}}$. Hence, wl_i^{D} is given by

$$wl_i^{\mathrm{D}} = \lim_{B'_i \uparrow B_i^{\mathrm{D}}} l'_i(B'_i). \tag{43}$$

Because $\frac{L'_i(B'_i)}{T_i}$ is a strictly increasing function of B'_i, we can use Lemma 17 in the following derivation

$$\begin{aligned}
\lim_{B'_i \uparrow B_i^{\mathrm{D}}} l'_i(B'_i) &= \lim_{B'_i \uparrow B_i^{\mathrm{D}}} \left\lceil \frac{L'_i(B'_i)}{T_i} \right\rceil \\
&= \{\text{Lemma 17}\} \left\lceil \lim_{B'_i \uparrow B_i^{\mathrm{D}}} \frac{L'_i(B'_i)}{T_i} \right\rceil \\
&= \{(41)\} \left\lceil \frac{WL_i^{\mathrm{D}}}{T_i} \right\rceil.
\end{aligned}$$

Equation (42) immediately follows from Equation (43) and this latter equation.

The proof that wl_i^{D} is a maximum consists of two steps. We first prove that $l'_i(B'_i)$ is left-continuous in B_i^{D}, i.e.

$$l'_i(B_i^{\mathrm{D}}) = \lim_{B'_i \uparrow B_i^{\mathrm{D}}} l'_i(B'_i), \tag{44}$$

and subsequently prove that $l'_i(B'_i)$ is constant in an interval $(B_i^{\mathrm{D}} - \gamma, B_i^{\mathrm{D}}]$ for a sufficiently small $\gamma \in \mathbb{R}^+$, i.e.

$$\forall_{B_i^{\mathrm{D}} - \gamma < B'_i \leq B_i^{\mathrm{D}}} \; l'_i(B'_i) = wl_i^{\mathrm{D}}.$$

To prove that $l'_i(B'_i)$ is left-continuous in B_i^{D}, we show that $L'_i(B_i^{\mathrm{D}})$ is defined and equal to WL_i^{D}, and subsequently show that $l'_i(B_i^{\mathrm{D}}) = wl_i^{\mathrm{D}}$. From Theorem 7, we know that $L'_i(B'_i)$ exists if Assumption 1 holds. Moreover, considering Theorem 6 and Lemma 5, we conclude that WL_i^{D} and $L'_i(B_i^{\mathrm{D}})$ are solutions of the same equation, i.e. $L'_i(B_i^{\mathrm{D}}) = WL_i^{\mathrm{D}}$. As a result, we get

$$l'_i(B_i^{\mathrm{D}}) = \left\lceil \frac{L'_i(B_i^{\mathrm{D}})}{T_i} \right\rceil = \left\lceil \frac{WL_i^{\mathrm{D}}}{T_i} \right\rceil = wl_i^{\mathrm{D}}.$$

To prove that $l'_i(B'_i)$ is constant in an interval $(B_i^{\mathrm{D}} - \gamma, B_i^{\mathrm{D}}]$ for a sufficiently small $\gamma \in \mathbb{R}^+$, we use the definition of a limit:

$$\lim_{x \uparrow X} f(x) = Y \Leftrightarrow \forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{x \in (X - \delta, X)} |f(x) - Y| < \varepsilon.$$

Because $l'_i(B'_i)$ is strictly non-decreasing and defined in B_i^{D}, we have

$$\forall_{0 \leq B'_i \leq B_i^{\mathrm{D}}} \; l'_i(B'_i) \leq wl_i^{\mathrm{D}}.$$

Let $\varepsilon \in (0,1]$. Now there exists a $\delta \in (0, B_i^{\mathrm{D}})$ such that $0 \leq wl_i^{\mathrm{D}} - l'_i(B'_i) < \varepsilon \leq 1$ for all $B'_i \in (B_i^{\mathrm{D}} - \delta, B_i^{\mathrm{D}}]$, hence $wl_i^{\mathrm{D}} \geq l'_i(B'_i) > wl_i^{\mathrm{D}} - 1$. Because $wl_i^{\mathrm{D}}, l'_i(B'_i) \in \mathbb{N}$, this completes the proof. □

Note that unlike WL_i^{D}, the value for wl_i^{D} can be assumed. Based on Lemma 6, we conclude that $l'_i((B_i^{\mathrm{D}} - \gamma)^+) = wl_i^{\mathrm{D}}$ for a sufficiently small $\gamma \in \mathbb{R}^+$, and we can therefore exchange the order of the operators in Equation (35), i.e.

$$WR_i^{\mathrm{D}} = \max_{0 \leq k < wl_i^{\mathrm{D}}} \lim_{\varepsilon \downarrow 0} r'_{ik}\left(\left(B_i^{\mathrm{D}} - \varepsilon\right)^+\right). \tag{45}$$

Hence, WR_{ik}^{D} is given by

$$WR_{ik}^{\mathrm{D}} = \lim_{\varepsilon \downarrow 0} r'_{ik}\left(\left(B_i^{\mathrm{D}} - \varepsilon\right)^+\right). \tag{46}$$

Lemma 7. *The worst-case response time* WR_{ik}^{D} *of job k with* $0 \leq k < wl_i^{\mathrm{D}}$ *of a task* τ_i *under FPDS and arbitrary phasing is given by*

$$WR_{ik}^{\mathrm{D}} = \begin{cases} WR_i^{\mathrm{P}}(B_i^{\mathrm{D}} + (k+1)C_i - F_i) + F_i - kT_i & \text{for } i < n \\ WO_n^{\mathrm{P}}((k+1)C_n - F_n) + F_n - kT_n & \text{for } i = n \end{cases}, \tag{47}$$

where $WR_i^{\mathrm{P}}(B_i^{\mathrm{D}} + (k+1)C_i - F_i)$ *and* $WO_i^{\mathrm{P}}(B_i^{\mathrm{D}} + (k+1)C_i - F_i)$ *are the worst-case response time and the worst-case occupied time under FPPS of a task* τ_i' *with a computation time* $C_i' = B_i^{\mathrm{D}} + (k+1)C_i - F_i$*, a period* $T_i' = kT_i + D_i - F_i$ *and a deadline* $D_i' = T_i'$*.*

Proof. Starting from Equation (46), we derive

$$\begin{aligned} WR_{ik}^{\mathrm{D}} &= \lim_{\varepsilon \downarrow 0} r_{ik}'\left(\left(B_i^{\mathrm{D}} - \varepsilon\right)^+\right) \\ &= \{(33)\} \lim_{\varepsilon \downarrow 0} \left(b_{ik,m_i}'\left(\left(B_i^{\mathrm{D}} - \varepsilon\right)^+\right) + F_i - kT_i\right) \\ &= \lim_{\varepsilon \downarrow 0} b_{ik,m_i}'\left(\left(B_i^{\mathrm{D}} - \varepsilon\right)^+\right) + F_i - kT_i, \end{aligned}$$

where $b_{ik,m_i}'\left(\left(B_i^{\mathrm{D}} - \varepsilon\right)^+\right)$ denotes the relative begin time of the final subjob of job k of task τ_i with $0 \leq k < wl_i$ and $\varphi_j' = 0$ for $j \leq i$ as given in Equation (34). Hence, $b_{ik,m_i}'\left(\left(B_i^{\mathrm{D}} - \varepsilon\right)^+\right)$ is the smallest $x \in \mathbb{R}^+$ satisfying

$$x = \left(\left(B_i^{\mathrm{D}} - \varepsilon\right)^+\right) + (k+1)C_i - F_i + \sum_{j<i} \left(\left\lfloor \frac{x}{T_j} \right\rfloor + 1\right) C_j.$$

Now let task set $\mathcal{T}'$ be identical to $\mathcal{T}$ except for the characteristics of task τ_i, i.e. τ_i' has characteristics $C_i' = \left(B_i^{\mathrm{D}} - \varepsilon\right)^+ + (k+1)C_i - F_i$, $T_i' = kT_i + D_i - F_i$, and $D_i' = T_i'$. Hence, task τ_i' of $\mathcal{T}'$ misses its deadline under FPPS and arbitrary phasing if and only if job k of task τ_i of $\mathcal{T}$ misses its deadline under FPDS, and arbitrary phasing and an amount of blocking $\left(B_i^{\mathrm{D}} - \varepsilon\right)^+$. Based on Theorem 4, we can now write

$$b_{ik,m_i}'\left(\left(B_i^{\mathrm{D}} - \varepsilon\right)^+\right) = WO_i^{\mathrm{P}}\left(\left(B_i^{\mathrm{D}} - \varepsilon\right)^+ + (k+1)C_i - F_i\right).$$

For $i = n$, we substitute $B_n^{\mathrm{D}} = 0$, and immediately arrive at Equation (47), which proves the lemma for $i = n$.

For $i < n$, we derive

$$\begin{aligned} WR_{ik}^{\mathrm{D}} &= \lim_{\varepsilon \downarrow 0} WO_i^{\mathrm{P}}\left(\left(B_i^{\mathrm{D}} - \varepsilon\right)^+ + (k+1)C_i - F_i\right) + F_i - kT_i \\ &= \{(14)\}\ WR_i^{\mathrm{P}}\left(B_i^{\mathrm{D}} + (k+1)C_i - F_i\right) + F_i - kT_i, \end{aligned}$$

which proves the lemma for $i < n$. □

Note that because the lowest priority task is the only task that cannot be blocked, i.e. $B_n^{\mathrm{D}} = 0$, the worst-case response time analysis for FPDS is not uniform for all tasks. Moreover, note that WR_{ik}^{D} is a supremum (and not a maximum) for all tasks, except for the lowest priority task, i.e. that value cannot be assumed for $i < n$.

Theorem 12. *The worst-case response time* WR_i^{D} *of a task* τ_i *under FPDS and arbitrary phasing is given by*

$$WR_i^{\mathrm{D}} = \max_{0 \leq k < wl_i^{\mathrm{D}}} WR_{ik}^{\mathrm{D}}. \tag{48}$$

Proof. The theorem follows immediately from Equations (45) and (46), and requires Lemma 7. □

6.3 An iterative procedure

The next theorem provides an iterative procedure to determine the worst-case response time WR_i^{D} for task τ_i under FPDS and arbitrary phasing. The procedure is stopped when the worst-case response time WR_{ik}^{D} of job k for task τ_i exceeds the deadline D_i or when the level-i active period is over. This latter condition is based on a property of WL_i^{D}.

Lemma 8. *The worst-case length WL_{ik}^{D} of a level-(i,k) active period under FPDS is the smallest positive $x \in \mathbb{R}^+$ satisfying the following equation*

$$x = B_i^{\mathrm{D}} + (k+1)C_i + \sum_{j<i} \left\lceil \frac{x}{T_j} \right\rceil C_j. \tag{49}$$

Proof. The proof is similar to the proof of Lemma 5. □

Note that because B_i^{D} is a supremum (and not a maximum) for all tasks, except the lowest priority task, WL_{ik}^{D} is also supremum (and not a maximum) for all tasks, except the lowest priority task, i.e. that value cannot be assumed for $i < n$.

Lemma 9. *The worst-case length WL_{ik}^{D} of a level-(i,k) active period under FPDS is given by*

$$WL_{ik}^{\mathrm{D}} = WR_i^{\mathrm{P}}(B_i^{\mathrm{D}} + (k+1)C_i). \tag{50}$$

where $WR_i^{\mathrm{P}}(B_i^{\mathrm{D}} + (k+1)C_i)$ is the worst-case response time under FPPS and arbitrary phasing of a task τ_i' with a computation time $C_i' = B_i^{\mathrm{D}} + (k+1)C_i$, a period $T_i' = (k+1)T_i + D_i$ and a deadline $D_i' = T_i'$.

Proof. The lemma follows from the similarity between Equations (7) and (49). The period and deadline of task τ_i' have been chosen to be equal to the deadline of job $k+1$ of task τ_i. Hence, when the iterative procedure to determine $WR_i^{\mathrm{P}}\left(B_i^{\mathrm{D}} + (k+1)C_i\right)$ stops because the deadline D_i' is exceeded, the deadline $d_{i,k+1}$ will be exceeded as well. □

Lemma 10. *Let $k' \in \mathbb{N}$ be the smallest value for which $WR_i^{\mathrm{P}}\left(B_i^{\mathrm{D}} + (k'+1)C_i\right) \leq (k'+1)T_i$. The worst-case length WL_i^{D} of a level-i active period is now given by $WR_i^{\mathrm{P}}\left(B_i^{\mathrm{D}} + (k'+1)C_i\right)$.*

Proof. To prove the lemma, we will prove the following equivalent relation by means of a contradiction argument

$$\underset{0 \leq k < wl_i^{\mathrm{D}}}{\forall} \left(WL_{ik}^{\mathrm{D}} \leq (k+1)T_i \Rightarrow k = wl_i^{\mathrm{D}} - 1 \right).$$

We only consider $k < wl_i^{\mathrm{D}} - 1$, because the proof for $k = wl_i^{\mathrm{D}} - 1$ is similar.

Let $WL_{i,k}^{\mathrm{D}} \leq (k+1)T_i$ for $0 \leq k < wl_i^{\mathrm{D}} - 1$. Using Lemma 9, we derive $WR_i^{\mathrm{P}}(B_i^{\mathrm{D}} + (k+1)C_i) \leq (k+1)T_i$. Hence, task τ_i' has a completion at or before $(k+1)T_i$, and all higher priority tasks that are released in the interval $[0, WR_i^{\mathrm{P}}\left(B_i^{\mathrm{D}} + (k+1)C_i\right))$ have a completion in that interval. Because task τ_i' represents the executions of both the blocking lower priority task as well as task τ_i, all executions of the corresponding jobs also have a completion in that interval. Hence, the level-i active period that started with an ε-critical instant ends at time $WR_i^{\mathrm{P}}\left(B_i^{\mathrm{D}} + (k+1)C_i\right)$. However, we now have that the length of the level-i active period equals $WL_{i,k}^{\mathrm{D}}$, a value that is strictly smaller than WL_i^{D}, which is a contradiction. Therefore, our assumption that $WL_{i,k}^{\mathrm{D}} \leq (k+1)T_i$ for $0 \leq k < wl_i^{\mathrm{D}} - 1$ is wrong, which proves the lemma. □

From these lemmas, we draw the following conclusion.

Corollary 7. *The level-i active period is over for the smallest $k' \in \mathbb{N}$ for which $WR_i^{\mathrm{P}}(B_i^{\mathrm{D}} + (k'+1)C_i) \leq (k'+1)T_i$.* □

Theorem 13. *The worst-case response time WR_i^{D} of a task τ_i can be found by the following iterative procedure under Assumption 1, using (47).*

$$WR_i^{(0)} = WR_{i,0}^{\mathrm{D}} \tag{51}$$

$$WR_i^{(l+1)} = \max(WR_i^{(l)}, WR_{i,l+1}^{\mathrm{D}}) \quad l = 0,1,\ldots \tag{52}$$

The procedure is stopped when the worst-case response time WR_{ik}^{D} of job k of task τ_i exceeds the deadline D_i or when the level-i active period is over, i.e. $WR_i^{\mathrm{P}}(B_i^{\mathrm{D}} + (k+1)C_i) \leq (k+1)T_i$.

Proof. Corollary 7 states that $WR_i^{\mathrm{P}}(B_i^{\mathrm{D}} + (k+1)C_i) \leq (k+1)T_i$ is a proper termination condition to determine whether or not the level-i active period is over before the release of job $k+1$. Because of Theorem 7, the level-i active periods ends under Assumption 1, and we therefore have to consider at most a finite number wl_i^{D} of jobs of task τ_i. As a result, the iterative procedure ends. We observe that the iterative procedure also stops when the deadline D_i is exceeded, by the worst-case response time WR_{ik}^{D} of job k of τ_i i.e. when the task set is not schedulable. □

Corollary 8. *When Assumption 1 holds, we can derive the schedulability of a set of tasks $\mathcal{T}$ under FPDS and arbitrary phasing by checking the schedulability criterion $WR_i^{\mathrm{D}} \leq D_i$ using Theorem 13.* □

Corollary 9. *To check the schedulability criterion $WR_i^{\mathrm{D}} \leq D_i$ we do not need to determine the length WL_i^{D} of the worst-case level-i active period under FPDS first. Instead, we can simply check whether or not the level-i active period is over after every iteration.* □

Finally note that

- $WR_{i,k}^{\mathrm{D}}$ can be used as initial value to calculate $WR_i^{\mathrm{P}}(B_i^{\mathrm{D}} + (k+1)C_i)$ to determine whether or not the level-i active period is over before the release of job $k+1$;
- $WR_i^{\mathrm{P}}(B_i^{\mathrm{D}} + (k+1)C_i)$ can be used as initial value to calculate $WR_i^{\mathrm{P}}(B_i^{\mathrm{D}} + (k+2)C_i - F_i)$ to determine $WR_{i,k+1}^{\mathrm{D}}$.

7 Examples

In this section, we will illustrate the worst-case response time analysis presented in Section 6 to determine the schedulability of tasks and task sets under FPDS and arbitrary phasing of some examples of Section 4 using the iterative procedure presented in Theorem 13.

7.1 Schedulability of task τ_2 of $\mathcal{T}_2$

The schedulability of task τ_2 of task set $\mathcal{T}_2$ is the topic of this section. The characteristics of the tasks of $\mathcal{T}_2$ can be found in Table 2 on page 8 in Section 4.2.

To determine the worst-case response time WR_2^{D} for task τ_2, we first derive $B_2^{\mathrm{D}} = 2$ using Equation (17). Next, we determine $WR_2^{(0)}$ using Lemma 7, i.e.

$$WR_2^{(0)} = WR_{2,0}^{\mathrm{D}} = WR_2^{\mathrm{P}}(B_2^{\mathrm{D}} + C_2 - F_2) + F_2 = WR_2^{\mathrm{P}}(3) + 2 = 5 + 2 = 7.$$

Because $WR_{2,0}^{\mathrm{D}} \leq D_2 = 7$ and $WR_2^{\mathrm{P}}(B_2^{\mathrm{D}} + C_2) = WR_2^{\mathrm{P}}(5) = 9 > T_2 = 7$, i.e. the level-2 active period is not over yet, we proceed with the 2^{nd} job.

For the 2^{nd} job, we find

$$WR_{2,1}^{\mathrm{D}} = WR_2^{\mathrm{P}}(B_2^{\mathrm{D}} + 2C_2 - F_2) + F_2 - T_2 = WR_2^{\mathrm{P}}(6) - 5 = 10 - 5 = 5,$$

and therefore $WR_2^{(1)} = \max(WR_2^{(0)}, WR_{2,1}^{\mathrm{D}}) = \max(7,5) = 7$. Now $WR_{2,1}^{\mathrm{D}} = 5 \leq D_2$ and $WR_2^{\mathrm{P}}(B_2^{\mathrm{D}} + 2C_2) = WR_2^{\mathrm{P}}(8) = 14 \leq 2T_2 = 14$. Hence, we know that the level-2 active period is over, all jobs of task τ_2 meet their deadlines in that period, and the worst-case response time $WR_2^{\mathrm{D}} = 7$.

7.2 Schedulability of task τ_2 of $\mathcal{T}_4$

We will determine the schedulability of task τ_2 of task set $\mathcal{T}_4$ in this section. The characteristics of the tasks of $\mathcal{T}_4$ can be found in Table 4 on page 10 in Section 4.3.2.

We first determine $WR_2^{(0)}$ using Lemma 7, i.e.

$$WR_2^{(0)} = WR_{2,0}^{\mathrm{D}} = WO_2^{\mathrm{P}}(B_2^{\mathrm{D}} + C_2 - F_2) + F_2 = WO_2^{\mathrm{P}}(2) + 2.1 = 4 + 2.1 = 6.1.$$

Because $WR_{2,0}^{\mathrm{D}} \leq D_2 = 7$ and $WR_2^{\mathrm{P}}(B_2^{\mathrm{D}} + C_2) = WR_2^{\mathrm{P}}(4.1) = 8.1 > T_2 = 7$, we proceed with the 2^{nd} job.

For the 2^{nd} job, we find

$$WR_{2,1}^{\mathrm{D}} = WO_2^{\mathrm{P}}(B_2^{\mathrm{D}} + 2C_2 - F_2) + F_2 - T_2 = WO_2^{\mathrm{P}}(6.1) - 4.9 = 12.1 - 4.9 = 7.2.$$

Because $WR_{2,1}^{\mathrm{D}} > D_2 = 7$, we conclude that task τ_2 is not schedulable.

7.3 Schedulability of the task set $\mathcal{T}_5$

In this section, we will determine the schedulability of the task set $\mathcal{T}_5$. The characteristics of the tasks of $\mathcal{T}_5$ can be found in Table 5 on page 10 in Section 4.3.3.

To determine the worst-case response time WR_1^{D} for task τ_1, we first derive $B_1^{\mathrm{D}} = 3$ using Equation (17). Next, we determine $WR_2^{(0)}$ using Lemma 7, i.e.

$$WR_1^{(0)} = WR_{1,0}^{\mathrm{D}} = WR_1^{\mathrm{P}}(B_1^{\mathrm{D}} + C_1 - F_1) + F_1 = 3 + 2 = 5.$$

Now $WR_{1,0}^{\mathrm{D}} = D_1$ and $WR_1^{\mathrm{P}}(B_1^{\mathrm{D}} + C_1) = 5 = T_1$. Hence, we know that the level-1 active period is over, all jobs of task τ_1 meet their deadlines, and the worst-case response time $WR_1^{\mathrm{D}} = 5$.

Next, we determine the worst-case response time WR_2^{D} for task τ_2. To this end, we first determine $WR_2^{(0)}$ using Lemma 7, i.e.

$$WR_2^{(0)} = WR_{2,0}^{\mathrm{D}} = WO_2^{\mathrm{P}}(B_2^{\mathrm{D}} + C_2 - F_2) + F_2 = WO_2^{\mathrm{P}}(1.2) + 3 = 3.2 + 3 = 6.2.$$

Because $WR_{2,0}^{\mathrm{D}} < D_2 = 7$ and $WR_2^{\mathrm{P}}(B_2^{\mathrm{D}} + C_2) = 8.2 > T_2 = 7$, we proceed with the 2^{nd} job.

For the 2^{nd} job, we find

$$WR_{2,1}^{\mathrm{D}} = WO_2^{\mathrm{P}}(B_2^{\mathrm{D}} + 2C_2 - F_2) + F_2 - T_2 = WO_2^{\mathrm{P}}(5.4) - 4 = 9.4 - 4 = 5.4,$$

and therefore $WR_2^{(1)} = \max(WR_2^{(0)}, WR_{2,1}^{\mathrm{D}}) = \max(6.2, 5.4) = 6.2$. Because $WR_{2,1}^{\mathrm{D}} < D_2$ and $WR_2^{\mathrm{P}}(B_2^{\mathrm{D}} + 2C_2) = 14.4 > 2T_2 = 14$, we proceed with the 3^{rd} job.

For the 3^{rd} job, we find

$$WR_{2,2}^{\mathrm{D}} = WO_2^{\mathrm{P}}(B_2^{\mathrm{D}} + 3C_2 - F_2) + F_2 - 2T_2 = WO_2^{\mathrm{P}}(9.6) - 11 = 17.6 - 11 = 6.6,$$

and therefore $WR_2^{(2)} = \max(WR_2^{(1)}, WR_{2,2}^{\mathrm{D}}) = \max(6.2, 6.6) = 6.6$. Because $WR_{2,2}^{\mathrm{D}} < D_2$ and $WR_2^{\mathrm{P}}(B_2^{\mathrm{D}} + 3C_2) = 22.6 > 3T_2 = 21$, we proceed with the 4^{th} job.

For the 4^{th} job, we find

$$WR_{2,3}^{\mathrm{D}} = WO_2^{\mathrm{P}}(B_2^{\mathrm{D}} + 4C_2 - F_2) + F_2 - 3T_2 = WO_2^{\mathrm{P}}(13.8) - 18 = 23.8 - 18 = 5.8.$$

and therefore $WR_2^{(3)} = \max(WR_2^{(2)}, WR_{2,3}^{\mathrm{D}}) = \max(6.6, 5.8) = 6.6$. Because $WR_{2,3}^{\mathrm{D}} < D_2$ and $WR_2^{\mathrm{P}}(B_2^{\mathrm{D}} + 4C_2) = 28.8 > 4T_2 = 28$, we proceed with the 5^{th} job.

For the 5^{th} job, we find

$$WR_{2,4}^{\mathrm{D}} = WO_2^{\mathrm{P}}(B_2^{\mathrm{D}} + 5C_2 - F_2) + F_2 - 4T_2 = WO_2^{\mathrm{P}}(18) - 25 = 32 - 25 = 7,$$

and therefore $WR_2^{(4)} = \max(WR_2^{(3)}, WR_{2,4}^{\mathrm{D}}) = \max(6.6, 7) = 7$. Now $WR_{2,4}^{\mathrm{D}} = D_2$ and $WR_2^{\mathrm{P}}(B_2^{\mathrm{D}} + 5C_2) = 35 = 5T_2$. Hence we know that the level-2 active period is over, all jobs of task τ_2 meet their deadlines in that period, and the worst-case response time $WR_2^{\mathrm{D}} = 7$.

Because $WR_i^{\mathrm{D}} \leq D_i$ for all $i \leq n$, we conclude that $\mathcal{T}_5$ is schedulable under FPDS and arbitrary phasing when deadlines are equal to periods.

8 Discussion

This section presents a theorem for the worst-case response time of the highest priority task, compares the notion of level-*i* active period with similar notions in the literature, and presents pessimistic variants for the worst-case response time analysis of tasks under FPDS and arbitrary phasing.

8.1 Worst-case response time of highest priority task

In Section 4.4, we concluded that the optimism in the existing analysis does not occur for the highest priority task. The next theorem provides a formal basis for that conclusion, by stating that the worst-case response time of the highest priority task τ_1 can be found by only considering the first job of τ_1 in a level-1 active period started at an ε-critical instant.

First, we prove the following lemma.

Lemma 11. *The first job of task τ_1 in a level-1 active period has the largest response time of all jobs of τ_1 in that period.*

Proof. The highest priority task τ_1 experiences blocking of at most one subjob of a lower priority task. If the first job of τ_1 in a level-1 active period is blocked by an amount B, its response time $r'_{1,0}(B)$ becomes

$$r'_{1,0}(B) = B + C_1.$$

Now, assume the level-1 active period contains $l_1 > 1$ jobs of task τ_1. The response time $r'_{1,k}(B)$ of job k, with $0 \leq k < l_1$, becomes

$$\begin{aligned} r'_{1,k}(B) &= B + (k+1)C_1 - kT_1 \\ &= B + C_1 + k(C_1 - T_1) \\ &= B + C_1 + k(U_1 - 1)T_1 \end{aligned}$$

When task τ_1 is blocked by a lower priority task, $U_1 < 1$. Hence, we find

$$r'_{1,k}(B) \quad < \quad B + C_1 = r'_{1,0}(B),$$

which proves the lemma. □

Theorem 14. *The worst-case response time WR_1^{D} of the highest priority task τ_1 under FPDS is equal to*

$$WR_1^{\mathrm{D}} = B_1^{\mathrm{D}} + C_1. \tag{53}$$

Proof. From equation $r'_{1,0}(B) = B + C_1$, we conclude that $r'_{1,0}(B)$ is a strictly increasing function of B. Hence, we derive

$$WR_1^{\mathrm{D}} = \sup_B r'_{1,0}(B) = \lim_{B \uparrow B_1^{\mathrm{D}}} (B + C_1) = B_1^{\mathrm{D}} + C_1,$$

which proves the theorem. □

8.2 A comparison with existing notions

We will now compare our notion of level-*i* active period with similar notions in the literature.

8.2.1 Level-*i* busy period in [27]

The notion of *level-i busy period* originates from [27], where it has been introduced as an expedient to determine the worst-case response times of tasks for deadlines larger than periods under FPPS and arbitrary phasing. The level-*i* busy period is defined as follows.

Definition 11. *A level-i busy period is a time interval $[a,b]$ within which jobs of priority i or higher are processed throughout $[a,b]$ but no jobs of level i or higher are processed in $(a-\varepsilon,a)$ or $(b,b+\varepsilon)$ for sufficiently small $\varepsilon > 0$.* □

Figure 9 also shows the level-1 busy periods and level-2 busy periods for $\mathcal{T}_1$. The level-1 busy periods in this figure only differ from the level-1 active periods by the inclusion of the end-points of the intervals by the former. The difference between level-2 busy periods and level-2 active periods is more significant, however. Whereas the interval $[0,12)$ is constituted by four level-2 active periods, i.e. $[0,5)$, $[5,7)$, $[7,10)$, and $[10,12)$, the interval is contained in a single level-2 busy period $[0,12]$. Stated in other words, the level-2 busy period unifies four adjacent level-2 active periods. Similarly, the interval $[20,27)$ is constituted by two level-2 active periods, i.e. $[20,25)$ and $[25,27)$, and the interval is contained in a single level-2 busy period $[20,27]$.

Figure 10 shows the level-1 busy periods and level-2 busy periods for $\mathcal{T}_1$. From this figure, we see that the level-2 busy period never ends for $U = 1$, as also becomes immediately clear from Definition 11. Conversely, the level-2 active period that started at time $t = 0$ ends at time $t = 35$; see also Assumption 1 and Theorem 7. We observe that the definition of level-*i* busy period is included in [24] (on page D-4, referring to [27]), and the notion is used in Technique 5 "Calculating Response Time with Arbitrary Deadlines and Blocking." As shown above, the busy period never ends for $U = 1$. Notably, in [24] on page 4–35 it is only mentioned that *we must be sure that the [...] utilization [...] is not greater than one*. In Step 6 "Decide if the busy period is over" the notion is used to determine whether or not the iterative procedure can be stopped. Notably, that decision is not based on the end of the busy period, but on the end of the level-*i* active period, i.e. when the (worst-case) response time WR^{P}_{ik} of job k of task τ_i is less than or equal to T_i; see also Theorem 13.

There is another striking difference between the level-*i* active period and the level-*i* busy period. A level-*i* active period may start when a task with a lower priority is still being processed, as illustrated by the level-1 active period that starts at time $t = 5$ in Figure 10. The corresponding level-1 busy period does not start at time $t = 5$, but at time $t = 6.2$ instead.

The fundamental difference between both notions can be traced back to their definitions; a busy period is based on a schedule, i.e. the definition refers to processing of jobs, whereas an active period is based on (pending) load or active jobs.

8.2.2 τ_i-busy period in [19]

In [19], the notion of busy period is slightly modified to accommodate the fact that a task τ_i may be composed of distinct subtasks, each of which may have its own timing requirements and fixed priority. In the following definition, ρ_i denotes the minimum priority of the subtasks of task τ_i.

Definition 12. *A* τ_i-idle instant *is any time* t *such that all work of priority* ρ_i *or higher started before* t *and all* τ_i *jobs also started before* t *have completed at or before* t. ☐

Definition 13. *A* τ_i-busy period *is an interval of time* $[A,B]$ *such that both* A *and* B *are* τ_i*-idle instants and there is no time* $t \in (A,B)$ *such that* t *is a* τ_i*-idle instant.* ☐

This notion of τ_i-busy period is similar to our level-*i* active period, with as main difference that a τ_i-busy period is a closed interval rather than a right semi-open interval. Although this difference may be viewed as philosophical, we prefer the usage of a right semi-open interval, which we will motivate by means of Figure 10. Given Definition 12 and 13, time $t = 35$ belongs to two τ_2-busy periods, i.e. $[0,35]$ and $[35,70]$. Moreover, time $t = 35$ is also a τ_2-idle instant. Hence, τ_i-busy periods can overlap, and when they overlap, the overlap is termed a τ_i-idle instant. This is considered to be counter-intuitive.

8.2.3 Level-*i* busy period in [18]

After a brief recapitulation of the notion of level-*i* busy period of [27] for FPPS, an informal description of a level-*i* busy period for FPNS under discrete scheduling [4] is given in Appendix A.2 of [18]. Note that for discrete scheduling, all task parameters are integers, i.e. T_i, C_i, $D_i \in \mathbb{Z}^+$ and $\varphi_i \in \mathbb{Z}^+ \cup \{0\}$, and preemptions are restricted to integer time points.

Unfortunately, there is an inconsistency in [18]. In Appendix A.2, the following definition is given.

Definition 14. *A* level-*i* busy period *is a processor busy period in which only instances of tasks with a priority greater than or equal to that of* τ_i *execute.* ☐

Accordingly, the interval of time that a lower priority task blocks task τ_i and its higher priority tasks is not included in the level-*i* busy period in both the text of the proof of Lemma 6 in Section 4.3.1 and Figure 6, which is used to illustrate that proof, Conversely, that interval *is* included in the equation to determine the length of the level-*i* busy period for the non-preemptive case, as described in Appendix A.2 in [18].

Note that [18] does not reproduce the definition of [27] (see Definition 11 above), but presents a new definition. Surprisingly, the differences between these definitions are not discussed. As an example, a (synchronous processor) busy period in [18] is described as a right semi-open interval on page 6, whereas the level-*i* busy period in [27] is a closed interval.

The notion of level-*i* busy period for FPNS in [18] is similar to our notion of level-*i* active period under the assumption that the equation to determine the length of a level-*i* busy period for the non-preemptive case properly reflects the intention of the authors.

8.2.4 Level-π_i busy interval in [30]

In [30], an analysis method is described to determine the schedulability of tasks under FPPS whose relative deadlines are larger than their respective periods, using the term *level-*π_i *busy interval*. A level-π_i busy interval is defined as a *left* semi-open interval $(t_0,t]$, i.e. the partitioning of the timeline in [30] differs from ours. Given the description in [30], our definition of level-*i* active period can be viewed as a slightly modified definition of level-π_i busy interval to accommodate our scheduling model for FPDS.

8.3 Pessimistic variants

Given Equation (47) in Lemma 7, we observe that the worst-case response time analysis is not uniform for all tasks. The analysis can be made uniform at the cost of potentially introducing pessimism. This section presents two lemmas with pessimistic variants for the worst-case response time analysis, one based on worst-case occupied times and one based on worst-case response times. For both variants, the iterative procedure presented in Theorem 13 can be used, i.e. only the equations for WR^{D}_{ik} change, not the iterative procedure. We conclude this section with a retrospect on the analysis for FPDS.

8.3.1 A uniform analysis based on WO^{P}

Lemma 12. *A pessimistic worst-case response time* $\widehat{WR}^{\mathrm{D}}_{ik}$ *of job k with* $0 \leq k < wl^{\mathrm{D}}_i$ *of a task* τ_i *under FPDS and arbitrary phasing is given by*

$$\widehat{WR}^{\mathrm{D}}_{ik} = WO^{\mathrm{P}}_i(B^{\mathrm{D}}_i + (k+1)C_i - F_i) + F_i - kT_i, \tag{54}$$

where $WO^{\mathrm{P}}_i(B^{\mathrm{D}}_i + (k+1)C_i - F_i)$ *is the worst-case occupied time under FPPS of a task* τ'_i *with a computation time* $C'_i = B^{\mathrm{D}}_i + (k+1)C_i - F_i$, *a period* $T'_i = kT_i + D_i - F_i$, *and a deadline* $D'_i = T'_i$.

Proof. By definition, $WR^{\mathrm{P}}_i(C) \leq WO^{\mathrm{P}}_i(C)$, hence $WR^{\mathrm{D}}_{ik} \leq \widehat{WR}^{\mathrm{D}}_{ik}$. Because $WR^{\mathrm{P}}_1(C) = WO^{\mathrm{P}}_1(C)$, $\widehat{WR}^{\mathrm{D}}_{ik}$ is potentially pessimistic for $1 < i < n$. □

The pessimism is illustrated by the set $\mathcal{T}_2$ consisting of three tasks with characteristics as described in Table 2 on page 8. For the worst-case response time $\widehat{WR}^{\mathrm{D}}_{2,0}$ of the first job of task τ_2 we find

$$\begin{aligned} \widehat{WR}^{\mathrm{D}}_{2,0} &= WO^{\mathrm{P}}_2(B^{\mathrm{D}}_2 + C_2 - F_2) + F_2 \\ &= WO^{\mathrm{P}}_2(2 + 3 - 2) + 2 \\ &= WO^{\mathrm{P}}_2(3) + 2 = 7 + 2 = 9. \end{aligned}$$

Because $\widehat{WR}^{\mathrm{D}}_{2,0} > D_2$, $\mathcal{T}_2$ is considered unschedulable under FPDS based on Lemma 12. Conversely, application of Lemma 7 yields a value $WR^{\mathrm{D}}_2 = 7 \leq D_2$.

We observe that $\widehat{WR}^{\mathrm{D}}_{2,0}$ is equal to $\widetilde{WR}^{\mathrm{D}}_2$ as determined in Section 4.2 by means of the existing analysis as presented in [12] and [15]. This equality is not a coincidence, for the following two reasons. Firstly, remember that because the characteristics of the tasks of $\mathcal{T}_2$ are integral multiples of a value $\delta = 1$ and $\Delta = 0.2 \leq \delta$, the value for $\widetilde{WR}^{\mathrm{D}}_2$ does not change when Δ is reduced to an arbitrary small positive value, i.e.

$$\widetilde{WR}^{\mathrm{D}}_2 = \lim_{\Delta \downarrow 0} \left(WR^{\mathrm{P}}_2(B^{\mathrm{D}}_2 + C_2 - (F_2 - \Delta)) + (F_2 - \Delta) \right).$$

Secondly, we can make the following derivation using Equation (10)

$$\begin{aligned} \lim_{\Delta \downarrow 0} \left(WR^{\mathrm{P}}_2(B^{\mathrm{D}}_2 + C_2 - (F_2 - \Delta)) + (F_2 - \Delta) \right) &= \lim_{\Delta \downarrow 0} \left(WR^{\mathrm{P}}_2(B^{\mathrm{D}}_2 + C_2 - (F_2 - \Delta)) \right) + F_2 \\ &= \{(10)\} \quad WO^{\mathrm{P}}_2(B^{\mathrm{D}}_2 + C_2 - F_2) + F_2 \\ &= \widehat{WR}^{\mathrm{D}}_{2,0}. \end{aligned}$$

These two results show that $\widehat{WR}^{\mathrm{D}}_{2,0} = \widetilde{WR}^{\mathrm{D}}_2$ for $\mathcal{T}_2$.

8.3.2 A uniform analysis based on WR^{P}

We will give another pessimistic approach that is uniform for all tasks, which assumes a small positive value Δ and is based on WR^{P}.

Lemma 13. *A pessimistic worst-case response time* $\widehat{\widehat{WR}}^{\mathrm{D}}_{ik}$ *of job k with* $0 \leq k < wl^{\mathrm{D}}_i$ *of a task* τ_i *under FPDS and arbitrary phasing is given by*

$$\widehat{\widehat{WR}}^{\mathrm{D}}_{ik} = WR^{\mathrm{P}}_i(B^{\mathrm{D}}_i + (k+1)C_i - (F_i - \Delta)) + (F_i - \Delta) - kT_i \tag{55}$$

where

(i) $WR^{\mathrm{P}}_i(B^{\mathrm{D}}_i + (k+1)C_i - (F_i - \Delta))$ *is the worst-case response time under FPPS of a task* τ'_i *with a computation time* $C'_i = B^{\mathrm{D}}_i + (k+1)C_i - (F_i - \Delta)$, *a period* $T'_i = kT_i + D_i - (F_i - \Delta)$, *and a deadline* $D'_i = T'_i$;

(ii) Δ *is a sufficiently small positive number.*

Proof. Because $WR^{\mathrm{P}}_1(C) = WO^{\mathrm{P}}_1(C) = C$, $\widehat{\widehat{WR}}^{\mathrm{D}}_{1,0} = \widehat{WR}^{\mathrm{D}}_{1,0} = WR^{\mathrm{D}}_1$. Hence, this approach is not pessimistic for $i = 1$. We will now prove that $WR^{\mathrm{P}}_i(C+\Delta) - \Delta \geq WO^{\mathrm{P}}_i(C)$ for $1 < i \leq n$. The potential additional pessimism introduced by Equation (55) now immediately follows from Lemma 12, i.e. $\widehat{\widehat{WR}}^{\mathrm{D}}_{ik} \geq \widehat{WR}^{\mathrm{D}}_{ik}$.

By definition, task τ_i can start executing an additional amount of computation time Δ after having executed an amount C at time $WO^{\mathrm{P}}_i(C)$. Because execution of that additional computation time Δ takes at least an amount of time Δ, we immediately get $WR^{\mathrm{P}}_i(C+\Delta) \geq WO^{\mathrm{P}}_i(C) + \Delta$, which proves the theorem. ☐

Based on Equation (10), we first conclude that both lemmas are similar for an arbitrary small positive value of Δ, i.e. $\lim_{\Delta\downarrow 0} \widehat{\widehat{WR}}^{\mathrm{D}}_{ik} = \widehat{WR}^{\mathrm{D}}_{ik}$. The additional pessimism potentially introduced by Lemma 13 is illustrated by the set $\mathcal{T}_7$ consisting of three tasks with characteristics as described in Table 7. For this example, the task characteristics are integral multiples of $\delta = 0.5$. For $\Delta = 0.6 > \delta$, we find $\widehat{\widehat{WR}}^{\mathrm{D}}_{2,0} = 12$, which is larger than τ_2's deadline. Conversely, the worst-case response time $\widehat{WR}^{\mathrm{D}}_2$ of task τ_2 determined by means of Theorem 13 using Lemma 12 yields $\widehat{WR}^{\mathrm{D}}_2 = WR^{\mathrm{D}}_2 = 9 \leq D_2$. For $\Delta = 0.4 < \delta$, we find $\widehat{\widehat{WR}}^{\mathrm{D}}_{2,0} = 9$. For this value of Δ, $\widehat{\widehat{WR}}^{\mathrm{D}}_{2,0} = \widehat{WR}^{\mathrm{D}}_2 = WR^{\mathrm{D}}_2 = 9 \leq D_2$, and reducing the value of Δ will not change the value found for $\widehat{\widehat{WR}}^{\mathrm{D}}_{2,0}$.

	$T_i = D_i$	C_i
τ_1	6.5	3
τ_2	9	3
τ_3	30	3

Table 7. Task characteristics of $\mathcal{T}_7$.

The next lemma provides a sufficient condition to guarantee that Lemma 13 introduces no additional pessimism compared to Lemma 12.

Lemma 14. *If the greatest common divisor* ($\mathrm{gcd}^{\mathbb{R}^+}$) *of the periods and computation times of the tasks exists, and is equal to* δ, *then* $\Delta < \delta$ *is a sufficient condition to guarantee that Lemma 13 introduces no additional pessimism compared to Lemma 12.*

Proof. To prove the lemma, it suffices to prove

$$\Delta < \delta \Rightarrow WR^{\mathrm{P}}_i(B^{\mathrm{D}}_i + (k+1)C_i - (F_i - \Delta)) - \Delta = WO^{\mathrm{P}}_i(B^{\mathrm{D}}_i + (k+1)C_i - F_i).$$

From Theorem 2, we derive that $WR^{\mathrm{P}}_i(B^{\mathrm{D}}_i + (k+1)C_i - (F_i - \Delta))$ is given by the smallest $x \in \mathbb{R}^+$ that satisfies the following equation, provided that x is at most $kT_i + D_i - (F_i - \Delta)$,

$$x = B^{\mathrm{D}}_i + (k+1)C_i - (F_i - \Delta) + \sum_{j<i} \left\lceil \frac{x}{T_j} \right\rceil C_j.$$

By substituting $x = x' + \Delta$, we get

$$x' = B^{\mathrm{D}}_i + (k+1)C_i - F_i + \sum_{j<i} \left\lceil \frac{x' + \Delta}{T_j} \right\rceil C_j.$$

When the greatest common divisor ($\mathrm{gcd}^{\mathbb{R}^+}$) of the periods and computation times of the tasks exists and is equal to δ, all task parameters are integral multiples of δ (by definition), and x' will also be an integral multiple of δ. Let $x' = n_{x'} \cdot \delta$ and $T_j = n_{T_j} \cdot \delta$ for an arbitrary $j < i$, where $n_{x'}, n_{T_j} \in \mathbb{N}^+$. Now we get

$$\left\lceil \frac{x' + \Delta}{T_j} \right\rceil = \left\lceil \frac{n_{x'} + \frac{\Delta}{\delta}}{n_{T_j}} \right\rceil.$$

Moreover,

$$0 < \frac{\Delta}{\delta} < 1 \Rightarrow \left\lceil \frac{n_{x'} + \frac{\Delta}{\delta}}{n_{T_j}} \right\rceil = \left\lfloor \frac{n_{x'}}{n_{T_j}} \right\rfloor + 1.$$

Hence, if the gcd$^{\mathbb{R}^+}$ exists and is equal to $\delta > \Delta$, the smallest $x' \in \mathbb{R}^+$ satisfying the recursive equation given above is a solution for both $WR_i^{\mathrm{P}}(B_i^{\mathrm{D}} + (k+1)C_i - (F_i - \Delta)) - \Delta$ and $WO_i^{\mathrm{P}}(B_i^{\mathrm{D}} + (k+1)C_i - F_i)$, which proves the lemma. □

We finally observe that the analysis presented in Lemma 13 is similar to the revised schedulability analysis for CAN presented in [17]. The latter analysis is an evolutionary improvement of the analysis given by Tindell in [38, 37, 39]. A fixed value for Δ is used in [17], corresponding to the transmission time for a single bit on CAN .

8.3.3 A retrospect

Using our notation, the worst-case response time of a task τ_i under FPDS, arbitrary phasing, and deadlines less than or equal to periods, as described in [30] can be given by $WR_i^{\mathrm{P}}(B_i^{\mathrm{D}} + C_i)$. As observed in [14], this analysis is pessimistic, because a task τ_i cannot be preempted while executing its last subjob, i.e. F_i. The original improvement of the worst-case response time of a task τ_i under FPDS as presented in [14] was not based on B_i^{D} as given in Equation (17), but on the *maximum length of deferred preemption*, i.e. a blocking time $\widehat{\widehat{B}}$ given by

$$\widehat{\widehat{B}} = \max_{1 \le j \le n} \max_{1 \le k \le m_j} C_{j,k}. \tag{56}$$

Though pessimistic, this original improvement is correct, i.e. not optimistic. The problem with the analysis in [12, 15] is caused by the fact that the non-preemptive behavior of the final subjob of task τ_i itself is not taken into account, as illustrated by Figure 7 on page 11. As described in [17] in the context of schedulability analysis for CAN, this problem can therefore be resolved at the cost of potentially introducing additional pessimism by using $\widehat{B}_i^{\mathrm{D}}$, which is given by

$$\widehat{B}_i^{\mathrm{D}} = \max(B_i^{\mathrm{D}}, F_i). \tag{57}$$

Conversely, the problem with the analysis in [12, 15] does not occur when $\widehat{B}_i^{\mathrm{D}} = B_i^{\mathrm{D}}$, i.e. when $B_i^{\mathrm{D}} \ge F_i$.

8.4 An advanced model for FPDS

The model for FPDS described in Section 2.2 assumes that each job of a task τ_i consists of a sequence of m_i subjobs. In this section, we will illustrate by means of an example how our analytical results can be applied in a context where a task τ_i consists of a (rooted and connected) directed acyclic graph (DAG) of m_i subjobs.

Consider Figure 11, with a DAG of subjobs representing the flow graph of task τ_i. The nodes of this graph represent the subjobs and the edges represent the successor relationships of subjobs. The graph has a single root node, with a computation time of $C_{i,1}$, and two leaf nodes, with computation times $C_{i,7}$ and $C_{i,9}$, respectively. During the execution of a job, a single path from the root node to a leaf is traversed. Hence, a job will either execute the subjobs with computation times $C_{i,2}$ and $C_{i,3}$ or the subjob with computation time $C_{i,4}$. Similarly, a job will either execute $C_{i,6}$ and $C_{i,7}$ or $C_{i,8}$ and $C_{i,9}$. The structure of task τ_i plays a role during the analysis of the task itself, and for a lower priority task. The analysis of tasks with a higher priority than τ_i is similar to the case where a job consists of a sequence of subjobs. For the analysis of a task with a lower priority than τ_i, we need to determine the longest computation time C_i of τ_i for all possible paths through the graph. For our example, this is equal to

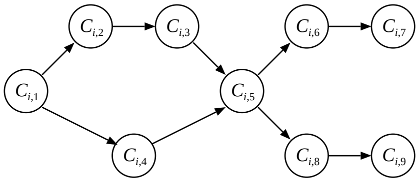

Figure 11. An example of a DAG of subjobs, representing the flow graph of task τ_i.

$$C_i = C_{i,1} + \max(C_{i,2} + C_{i,3}, C_{i,4}) + C_{i,5} + \max(C_{i,6} + C_{i,7}, C_{i,8} + C_{i,9}).$$

For the analysis of task τ_i itself, every leaf node of the DAG gives rise to a case that needs to be examined individually. For our example, we therefore get two cases, a first case for the leaf node $C_{i,7}$, i.e.

$$\begin{aligned} C_i' &= C_{i,1} + \max(C_{i,2} + C_{i,3}, C_{i,4}) + C_{i,5} + C_{i,6} + C_{i,7} \\ F_i' &= C_{i,7}, \end{aligned}$$

and a second case for the leaf node $C_{i,9}$, i.e.

$$\begin{aligned} C_i'' &= C_{i,1} + \max(C_{i,2} + C_{i,3}, C_{i,4}) + C_{i,5} + C_{i,8} + C_{i,9} \\ F_i'' &= C_{i,9}. \end{aligned}$$

The worst-case response time WR_i^{D} of task τ_i is the maximum of the worst-case response times of these two cases. Note that if $C_i' - F_i' \geq C_i'' - F_i''$ and $F_i' \geq F_i''$, then it suffices to consider the first case only. Similarly, if $C_i'' - F_i'' \geq C_i' - F_i'$ and $F_i'' \geq F_i'$, then it suffices to consider only the second case. As an alternative, we can also take a pessimistic approach, and determine WR_i^{D} based on

$$\begin{aligned} \widehat{C}_i &= \max(C_i' - F_i', C_i'' - F_i'') + \max(F_i', F_i'') \\ \widehat{F}_i &= \max(F_i', F_i''). \end{aligned}$$

We will now illustrate the analysis for τ_i with a numerical example. Consider the set $\mathcal{T}_8$ in Table 8. Assume a structure of each job of τ_2 as illustrated in Figure 11, and let the computation times of the subjobs of task τ_2 be given by $C_{2,1} = 1$, $C_{2,2} = 3$, $C_{2,3} = 4$, $C_{2,4} = 6$, $C_{2,5} = 1$, $C_{2,6} = 3$, $C_{2,7} = 2$, $C_{2,8} = 1$, $C_{2,9} = 5$. We now find $C_2' = 1 + \max(3+4, 6) + 1 + 3 + 2 = 14$, $F_2' = 2$, $C_2'' = 1 + \max(3+4, 6) + 1 + 1 + 5 = 15$, and $F_2'' = 5$. Because $C_2' - F_2' = 12 > C_2'' - F_2'' = 10$ and $F_2' = 2 < F_2'' = 5$, we have to determine the worst-case response times for both cases. Using the analysis presented in Section 6, we find 21 for the first case and 20 for the second case. The worst-case response time of τ_2 is therefore assumed for the first case, i.e. $WR_2^{\mathrm{D}} = 21$. For the pessimistic approach, we find $\widehat{C}_2 = \max(12, 10) + \max(2, 5) = 17$, $\widehat{F}_2 = 5$, and derive a worst-case response time for task τ_2 equal to 24.

	$T_i = D_i$	C_i
τ_1	16	2
τ_2	24	15
τ_3	36	3

Table 8. Task characteristics of $\mathcal{T}_8$.

9 Conclusions

In this paper, we revisited existing worst-case response time analysis of hard real-time tasks under FPDS, arbitrary phasing and relative deadlines at most equal to periods. We showed by means of a number of examples that existing analysis is pessimistic and/or optimistic, both for FPDS as well as for FPNS, being a special case of FPDS. From these examples, we concluded that the worst-case response time of a task is not necessarily assumed for the first job of a task when released at a critical instant. The reason for this is that the final subjob of a task can defer the execution of higher priority tasks, which can potentially give rise to higher interference for subsequent jobs of that task. This problem can therefore arise for all tasks, except for the highest priority task. We observed that González Harbour et al. [19] identified the same influence of jobs of a task for relative deadlines at most equal to periods in the context of FPPS of periodic tasks with varying execution priority.

We provided revised worst-case response time analysis, resolving the problems with existing approaches. The analysis is based on known concepts of critical instant and busy period for FPPS, for which we gave slightly modified definitions to accommodate for our scheduling model for FPDS. To prevent confusion with existing definitions of busy period, we used the term active period for our definition in this document. We discussed conditions for the termination of an active period, and presented a sufficient condition with a formal proof.

We showed that the critical instant, longest active period, and worst-case response time for a task are suprema rather than maxima for all tasks, except for the lowest priority task, i.e. that instant, period, and response time cannot be assumed. These anomalies for the lowest priority task are caused by the fact that only the lowest priority task cannot be blocked. We expressed worst-case response times under FPDS in terms of worst-case response times and worst-case occupied time under FPPS, and presented an iterative procedure to determine worst-case response times under FPDS.

We briefly compared the notion of level-i active period with similar notions in the literature. We concluded that the notions of τ_i-busy period in [19], level-i busy period in [18], and level-π_i busy interval in [30] are similar to our notion of level-i active period. There are striking differences with the notion of busy period in [27], however. In particular, the level-n busy period never ends for a utilization factor $U = 1$. Moreover, we observed that although [24] refers to the notion of busy period from [27] in their description of a method to determine worst-case response times of tasks under FPPS, arbitrary phasing and deadlines larger than periods, their termination condition is actually based on the notion of active period rather than busy period. We also presented uniform, but pessimistic variants of our worst-case response time analysis, and showed that the evolutionary improvement of the analysis for CAN as presented in [17] corresponds to one of these variants. Finally, we illustrated our analysis for a model for FPDS, where tasks are structured as flow graphs of subjobs rather than sequences.

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A Auxiliary definitions and lemmas

This appendix presents auxiliary definitions for greatest common divisor and least common multiple for both positive rational numbers and positive real numbers. Moreover, it presents auxiliary lemmas for a strictly increasing function $f(x)$.

Definition 15. *The* least common multiple for positive rational numbers *(*$\mathrm{lcm}^{\mathbb{Q}^+}$*) is defined as*

$$\mathrm{lcm}^{\mathbb{Q}^+}(r_1,\ldots,r_l) = \min\{r \in \mathbb{Q}^+ | r = n_1 \cdot r_1 = \ldots = n_l \cdot r_l \textit{ with } n_1,\ldots,n_l \in \mathbb{N}^+\}, \tag{58}$$

where $l \in \mathbb{N}$ *and* $l \geq 2$*, and* $r_1,\ldots,r_l \in \mathbb{Q}^+$. ☐

Definition 16. *The* greatest common divisor for positive rational numbers *(*$\mathrm{gcd}^{\mathbb{Q}^+}$*) is defined as*

$$\mathrm{gcd}^{\mathbb{Q}^+}(r_1,\ldots,r_l) = \max\{r \in \mathbb{Q}^+ | n_1 \cdot r = r_1,\ldots,n_l \cdot r = r_l \textit{ with } n_1,\ldots,n_l \in \mathbb{N}^+\}, \tag{59}$$

where $l \in \mathbb{N}$ *and* $l \geq 2$*, and* $r_1,\ldots,r_l \in \mathbb{Q}^+$. ☐

Definition 17. *The* least common multiple for positive real numbers *(*$\mathrm{lcm}^{\mathbb{R}^+}$*) is defined as*

$$\mathrm{lcm}^{\mathbb{R}^+}(r_1,\ldots,r_l) = \min\{r \in \mathbb{R}^+ | r = n_1 \cdot r_1 = \ldots = n_l \cdot r_l \textit{ with } n_1,\ldots,n_l \in \mathbb{N}^+\}, \tag{60}$$

where $l \in \mathbb{N}$ *and* $l \geq 2$*, and* $r_1,\ldots,r_l \in \mathbb{R}^+$. ☐

Definition 18. *The* greatest common divisor for positive real numbers *(*$\mathrm{gcd}^{\mathbb{R}^+}$*) is defined as*

$$\mathrm{gcd}^{\mathbb{R}^+}(r_1,\ldots,r_l) = \max\{r \in \mathbb{R}^+ | n_1 \cdot r = r_1,\ldots,n_l \cdot r = r_l \textit{ with } n_1,\ldots,n_l \in \mathbb{N}^+\}, \tag{61}$$

where $l \in \mathbb{N}$ *and* $l \geq 2$*, and* $r_1,\ldots,r_l \in \mathbb{R}^+$. ☐

Unlike $\mathrm{gcd}^{\mathbb{Q}^+}$ and $\mathrm{lcm}^{\mathbb{Q}^+}$, the greatest common divisor for positive real numbers $\mathrm{gcd}^{\mathbb{R}^+}$ and the least common multiple for positive real numbers $\mathrm{lcm}^{\mathbb{R}^+}$ need not exist.

Lemma 15 (Lemma 4.3 of [5]). *Let* $f(x)$ *be defined and strictly non-decreasing in an interval* $[a,b]$ *with* $f(a) > a$ *and* $f(b) < b$*. Then there exists a value* $c \in (a,b)$ *such that* $f(c) = c$*.*
Proof. See [5]. ☐

Lemma 16 (Lemma 4.5 in [5]). *When* $\lim_{x\downarrow X} f(x)$ *is defined, and* $f(x)$ *is strictly increasing in an interval* $(X, X+\gamma)$ *for sufficiently small* $\gamma \in \mathbb{R}^+$*, then the following equation holds.*

$$\lim_{x\downarrow X} \lceil f(x) \rceil = \left\lfloor \lim_{x\downarrow X} f(x) \right\rfloor + 1 \tag{62}$$

Proof. See [5]. ☐

Lemma 17. *When* $\lim_{x\uparrow X} f(x)$ *is defined, and f(x) is strictly increasing in an interval* $(X-\gamma, X)$ *for a sufficiently small* $\gamma \in \mathbb{R}^+$*, then the following equation holds.*

$$\lim_{x\uparrow X} \lceil f(x) \rceil = \left\lceil \lim_{x\uparrow X} f(x) \right\rceil \tag{63}$$

Proof. The proof uses the definition of limit:

$$\lim_{x\uparrow X} f(x) = Y \Leftrightarrow \underset{\varepsilon>0}{\forall} \underset{\delta>0}{\exists} \underset{x\in(X-\delta,X)}{\forall} |f(x) - Y| < \varepsilon.$$

We first prove the relation

$$\underset{X-\gamma<x<X}{\forall} f(x) < Y,$$

and subsequently prove the lemma.

The proof of the relation is based on a contradiction argument. Because $\lim_{x\uparrow X} f(x)$ is defined, we may write $\lim_{x\uparrow X} f(x) = Y$. Assume $f(x_1) \geq Y$ for an $x_1 \in (X-\gamma, X)$. Choose an $x_2 \in (x_1, X)$. Because $f(x)$ is strictly increasing in $(X-\gamma, X)$, $f(x_2) > f(x_1) \geq Y$. Now choose $\varepsilon = f(x_2) - Y$, then

$$\forall_{x \in (x_2, X)} f(x) > f(x_2) > Y$$

and hence

$$|f(x) - Y| > |f(x_2) - Y| = \varepsilon,$$

which contradicts the fact that $\lim_{x\uparrow X} f(x) = Y$.

For the proof of the lemma, we consider two main cases: $Y \in \mathbb{Z}$ and $Y \notin \mathbb{Z}$. Let $Y \in \mathbb{Z}$. According to the relation proved above, $0 < Y - f(x)$ for all $x \in (X-\gamma, X)$. Let $\varepsilon \in (0,1]$. Now there exists a $\delta_1 \in (0,\gamma)$ such that $0 < Y - f(x) < \varepsilon \leq 1$ for all $x \in (X-\delta_1, X)$, hence $Y > f(x) > Y-1$, i.e. $\lceil f(x) \rceil = Y = \lceil Y \rceil$. So,

$$\lim_{x\uparrow X} \lceil f(x) \rceil = \lim_{x\uparrow X} \lceil Y \rceil = \lceil Y \rceil = \left\lceil \lim_{x\uparrow X} f(x) \right\rceil .$$

Next, let $Y \notin \mathbb{Z}$. Let $\varepsilon \in (0, Y - \lfloor Y \rfloor]$. Now there exists a $\delta_2 \in (0,\gamma)$ such that for all $x \in (X-\delta_2, X)$

$$0 < Y - f(x) < \varepsilon \leq Y - \lfloor Y \rfloor ,$$

hence

$$Y > f(x) > Y - \varepsilon \geq \lfloor Y \rfloor ,$$

i.e.

$$\lceil f(x) \rceil = \lceil Y \rceil .$$

For this second main case we therefore also find

$$\lim_{x\uparrow X} \lceil f(x) \rceil = \lim_{x\uparrow X} \lceil Y \rceil = \lceil Y \rceil = \left\lceil \lim_{x\uparrow X} f(x) \right\rceil ,$$

which proves the lemma. □

The proofs of the following two lemmas are similar to the proofs of the previous two lemmas.

Lemma 18. *When* $\lim_{x\uparrow X} f(x)$ *is defined, and f(x) is strictly increasing in an interval* $(X-\gamma, X)$ *for a sufficiently small* $\gamma \in \mathbb{R}^+$*, then the following equation holds.*

$$\lim_{x\uparrow X} \lfloor f(x) \rfloor = \left\lceil \lim_{x\uparrow X} f(x) \right\rceil - 1 \tag{64}$$

□

Lemma 19. *When* $\lim_{x\downarrow X} f(x)$ *is defined, and* $f(x)$ *is strictly increasing in an interval* $(X, X+\gamma)$ *for sufficiently small* $\gamma \in \mathbb{R}^+$*, then the following equation holds.*

$$\lim_{x\downarrow X} \lfloor f(x) \rfloor = \left\lfloor \lim_{x\downarrow X} f(x) \right\rfloor \tag{65}$$

□

B On termination of a level-*n* active period

In this appendix, we give two examples of task sets with a utilization equal to 1 where the level-*n* active period does not end upon a simultaneous release of the tasks. For the first example, the least common multiple of the periods does not exist. Hence, the example shows that when Assumption 1 does not hold, the level-*n* active period need not end. The second example requires an extension of the scheduling model presented in Section 2 with activation (or release) jitter. For this extended model, it illustrates that even when the least common multiple of the periods exists, the level-*n* active period does not necessarily end for a processor utilization $U = 1$.

	T_i	C_i	φ_i
τ_1	2	1	0
τ_2	π	$\frac{\pi}{2}$	0

Table 9. Task characteristics of $\mathcal{T}_9$.

B.1 Least common multiple of the periods does not exist

Consider the task set $\mathcal{T}_9$ with task characteristics as given in Table 9. The utilization U of $\mathcal{T}_9$ is equal to $\frac{1}{2} + \frac{\pi/2}{\pi} = 1$. Because the ratio of the periods of the tasks is irrational, the least common multiple of the periods does not exist. We will now show that the following relation holds for the finalization time $f^{\mathrm{P}}_{2,k}$ of job k of task τ_2 under FPPS and a simultaneous release of τ_1 and τ_2 at time $t = 0$

$$(k+1)\pi < f^{\mathrm{P}}_{2,k} < (k+1)\pi + 1 \quad \text{for } k \geq 0. \tag{66}$$

Based on Corollary 7, we therefore conclude that the level-2 active period does not end. Now let $D_1 = T_1 = 2$ and $D_2 = \pi + 1$. Given Equation (66), we derive

$$T_2 = \pi < R^{\mathrm{P}}_{2,k} < \pi + 1 = D_2 \quad \text{for } k \geq 0,$$

and therefore know that the task set is schedulable under FPPS. However, if we try to determine whether or not the task set is schedulable under FPPS by means of the iterative procedure as described in, for example, [24], we find that the procedure does not terminate. This is because the termination condition of the procedure never holds, i.e. the response time of every job of task τ_2 is smaller than the deadline D_2, and larger than the period T_2.

We will now prove Equation (66). Task τ_1 is executing in the intervals $[lT_1, lT_1 + C_1) = [2l, 2l+1)$ for $l \in \mathbb{N}$, and the finalization time $f^{\mathrm{P}}_{2,k}$ of job k of task τ_2 is therefore in a complementary interval $[lT_1 + C_1, (l+1)T_1) = [2l+1, 2l+2)$. Let job k of τ_2 complete in the interval $[2m+1, 2m+2)$ for some $m \in \mathbb{N}$, i.e.

$$2m + 1 < f^{\mathrm{P}}_{2,k} \leq 2m + 2.$$

Because the utilization is 1 and we assume the tasks to be non-idling, there is no idle time in the interval $[0, f^{\mathrm{P}}_{2,k})$. Therefore, the interval $[0, f^{\mathrm{P}}_{2,k})$ contains exactly $m+1$ executions of task τ_1 and $k+1$ executions of task τ_2, i.e.

$$f^{\mathrm{P}}_{2,k} = (m+1)C_1 + (k+1)C_2 = (m+1) + (k+1)\frac{\pi}{2}.$$

Substituting this latter equation in the former relation yields

$$2m + 1 < (m+1) + (k+1)\frac{\pi}{2} \leq 2m + 2 \Leftrightarrow m < (k+1)\frac{\pi}{2} \leq m + 1.$$

Because $k, m \in \mathbb{N}$, we get

$$m + 1 > (k+1)\frac{\pi}{2}$$

and therefore

$$f^{\mathrm{P}}_{2,k} = (m+1) + (k+1)\frac{\pi}{2} > (k+1)\pi.$$

Moreover, because $m < (k+1)\frac{\pi}{2}$, we derive

$$f^{\mathrm{P}}_{2,k} = (m+1) + (k+1)\frac{\pi}{2} < (k+1)\pi + 1.$$

Together, these latter two relations for $f^{\mathrm{P}}_{2,k}$ prove Equation (66).

B.2 Activation jitter

With *activation* (or *release*) *jitter*, the releases of a task τ_i do not take place strictly periodically, with period T_i, but we assume they take place somewhere in an interval of length AJ_i that is repeated with period T_i. More specifically, the activations satisfy

$$\sup_{k,l} (a_{ik} - a_{il} - (k-l)T_i) \leq AJ_i. \tag{67}$$

	T_i	C_i	AJ_i
τ_1	4	2	1
τ_2	4	2	0

Table 10. Task characteristics of $\mathcal{T}_{10}$.

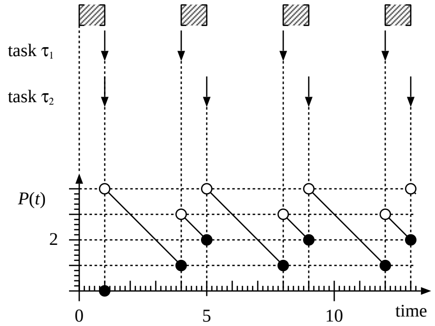


Figure 12. Activations for $\mathcal{T}_{10}$ and processor pending load $P(t)$.

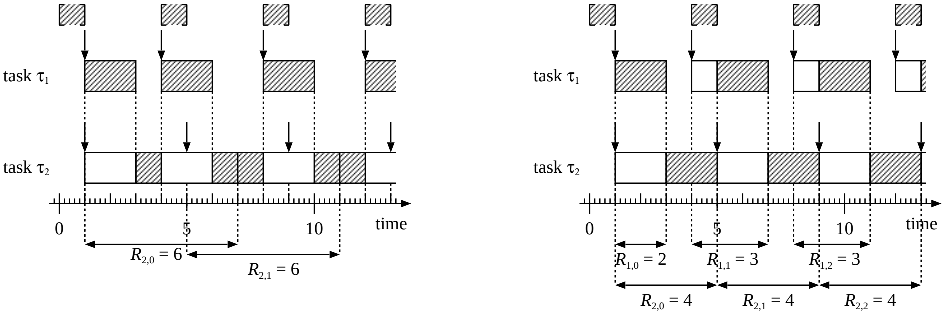
Consider task set $\mathcal{T}_{10}$ with task characteristics as given in Table 10. The least common multiple of the periods T_1 and T_2 is given by $\mathrm{lcm}(T_1, T_2) = 4$. Figure 12 shows the activations for task τ_1 and τ_2, with $a_{1,0} = AJ_1 = 1$, $a_{1l} = (l+1)T_1$ for $l \in \mathbb{N}$, and $a_{2,0} = 1$, and the processor pending load $P(t)$. These activations correspond to a critical instant for task τ_2 for FPPS and FPDS. For this example, the pending load is periodic, i.e. $P(t+4) = P(t)$ for $t > 1$. Because $P(t) > 0$ for $t > 1$, the level-2 active period never ends. As a consequence, the worst-case response time of τ_2 cannot be determined by means of an iterative procedure in which the response times of all activations in the level-2 active period are considered, irrespective of the scheduling algorithm. Hence, the common approach to determine the worst-case response time for τ_2 under FPPS, FPDS, and EDF [36] does not work.

Without proof, we merely state that the worst-case length WL_n of the level-n active period under arbitrary phasing and activation jitter is given by the smallest $x \in \mathbb{R}^+$ satisfying the following equation

$$x = \sum_{j \le n} \left\lceil \frac{x + AJ_j}{T_j} \right\rceil C_j,$$

where AJ_j is the activation jitter of task τ_j. As mentioned in [32], there exists a positive solution for this recursive equation if $U < 1$. The proof of this latter claim is similar to the proof of Lemma 2 on page 16.

Figure 13 shows timelines for $\mathcal{T}_{10}$ under FPPS, FPDS, and EDF. The figure illustrates that $\mathcal{T}_{10}$ is schedulable under FPDS and EDF for the given activations. Moreover, the schedule is periodic, i.e. $\sigma(t+4) = \sigma(t)$ for $t \ge 1$. $\mathcal{T}_{10}$ is also schedulable under FPPS when the deadline $D_2 \ge 6$ for task τ_2. Under FPPS, the schedule is also periodic, i.e. $\sigma(t+4) = \sigma(t)$ for $t \ge 3$. Because the schedule is periodic, the worst-case response time of task τ_2 can be determined by considering the response times of all jobs of τ_2 in a 'sufficiently long' interval, e.g. similar to the approach described in [28].



(a) Timeline under FPPS

(b) Timeline under EDF and FPDS

Figure 13. Timelines for $\mathcal{T}_{10}$ under FPPS, FPDS, and EDF with release jitter and a simultaneous release of both tasks at time $t = 1$.