Turbulent Functions and Solving the Navier-Stokes Equation by Fourier series

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Abstract: I give a resolution of the Navier-Stokes [2] equation by using the series of Fourier. Résumé: Je donne une résolution de l'équation de Navier-Stokes [2] par les séries de Fourier.

Keywords: Navier-Stokes, Fourier, Séries de Fourier.

I. INTRODUCTION

The Navier–Stokes equations is considered to be the first step to understanding the elusive phenomenon of turbulence, the Clay Mathematics Institute in May 2000 made this problem [2] one of its seven Millennium Prize problems in mathematics. In this article I will prove that the Navier-Stokes equation has a solutions and I will give techniques to resolve this beautiful equation. The Navier-Stokes equation, established in the nineteenth century by the French Navier and the British Stokes. It is an equation that describes the velocity field of a fluid. More specifically, it is a differential equation whose velocity field is unknown.

The Navier-Stokes equation is also used to predict the weather, the oceans simulate, optimize aircraft wings ... Knowing that a link between the Boltzmann equation and the Navier-Stokes equation was established, by studying the latter problem, I found that for to solve it we can reduce the problem of the heat-equation which is known can be solved by several methods : one of the first methods of solving the heat-equation was proposed by Joseph Fourier in his treatise analytical Theory of heat [1] in 1822. After giving a specific solution to the Navier-Stokes equation, I will demonstrate how to find all solutions of this equation if they exist, and I give the necessary and sufficient conditions for their existence. It will be seen in a remark that if the turbulence function is negligible, then the fluid will tend to behave like an ideal gas.

II. RECALL, NOTATIONS AND DEFINITIONS

Here are the Navier-Stokes equation:

$$\rho\left(\frac{\partial u}{\partial t}+(u.\ \nabla u)\right)=-\ \nabla p+\mu\ \nabla^2 u$$

$$\operatorname{div} u=0$$

Where u is the velocity field, p is the pressure, the density of the fluid, and μ its viscosity.

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And:

$$\nabla=\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right)$$

$$(u.\ \nabla u)=\sum_1^n u_i\frac{\partial u}{\partial x_i}$$

$$\nabla^2 u=\sum_1^n\frac{\partial^2 u}{\partial x_i^2}$$

$$\operatorname{div} u=\sum_1^n\frac{\partial u}{\partial x_i}$$

In the following, by dividing by ρ , the Navier-Stokes equation is of the form:

$$\frac{\partial u}{\partial t}+(u.\ \nabla u)=\alpha\ \nabla p+\beta\ \nabla^2 u$$

$$\operatorname{div} u=0$$

III. EXISTENCE OF THE SOLUTIONS FOR THE NAVIER-STOKES EQUATION:

On each axis i, try to find the solutions of the form:

$$\frac{\partial u_i}{\partial t}+\left(u_i\frac{\partial u_i}{\partial x_i}\right)=\alpha\frac{\partial p_i}{\partial x_i}+\beta\frac{\partial^2 u_i}{\partial x_i^2}$$

This is equivalent to:

$$\frac{\partial u_i}{\partial t}+\frac{\partial\left(\frac{1}{2}u_i^2-\alpha\, p_i\right)}{\partial x_i}=\beta\frac{\partial^2 u_i}{\partial x_i^2}$$

If u_i is a solution of the equation:

$$\frac{\partial u_i}{\partial t}=\beta\frac{\partial^2 u_i}{\partial x_i^2}$$

Such solutions u_i exist because the equation is analogous to the heat-equation which is resolvable by the Fourier series [1].

If p_i is such $\frac{1}{2}u_i^2-\alpha\, p_i=f_i(t)$, then

$$\frac{\partial\left(\frac{1}{2}u_i^2-\alpha\, p_i\right)}{\partial x_i}=0$$, and the equation is solved.

We do the same for all axes i until $i=n-1$.

For the axe $i=n$:

Let be $u_n=-\sum_1^{n-1}u_i$.

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We have: $\frac{\partial u_n}{\partial t}=\beta\frac{\partial^2 u_n}{\partial x_i^2}$, and if p_n is such $\frac{1}{2}u_n^2-\alpha\, p_n=f_n(t)$, then $\frac{\partial\left(\frac{1}{2}u_n^2-\alpha\, p_n\right)}{\partial x_n}=0$, and the equation is solved for the axe n.

It is clear that if e_i is the vector for the axes i, then $u=\sum_1^n u_i e_i$ is one solution of the Naviers-Stokes equation if $\operatorname{div} u=0$.

Else, to have $\operatorname{div} u=0$, we take u_i of the form:

$$u_i=e^{\beta t+\left(\sum_1^n x_i\right)},\ \forall i\in\{1,\dots,n-1\}$$

So we have solutions of the Navier-Stokes equation.

IV. NECESSARY CONDITIONS

Any solution (u,p) of the Navier-Stokes equation verifies that:

$$u_i^2-\alpha\, p_i=f_i(t),\ \forall i\in\{1,\dots,n\}$$

Indeed:

If (u,p) is a solution of the Navier-Stokes equation , we must have :

$$\frac{\partial u_i}{\partial t}+\frac{\partial\left(\frac{1}{2}u_i^2-\alpha\, p_i\right)}{\partial x_i}=\beta\frac{\partial^2 u_i}{\partial x_i^2}$$

Therefore:

$$-\frac{\partial u_i}{\partial t}=\frac{\partial\left(\frac{1}{2}u_i^2-\alpha\, p_i-\beta\frac{\partial u_i}{\partial x_i}\right)}{\partial x_i}$$

And:

$$-\partial x_i\frac{\partial u_i}{\partial t}=\partial\left(\frac{1}{2}u_i^2-\alpha\, p_i-\beta\frac{\partial u_i}{\partial x_i}\right)$$

When the fluid flows in one direction, then the space-time flows in the opposite direction with the same speed value:

We deduce that:

$$-u_i\partial u_i=\partial\left(\frac{1}{2}u_i^2-\alpha\, p_i-\beta\frac{\partial u_i}{\partial x_i}\right)$$

Therefore:

$$0=\partial\left(u_i^2-\alpha\, p_i-\beta\frac{\partial u_i}{\partial x_i}\right)$$

And:

$$u_i^2-\alpha\, p_i-\beta\frac{\partial u_i}{\partial x_i}=f_i(t)$$

So:

$$\sum_1^n u_i^2-\alpha\, p_i-\beta\frac{\partial u_i}{\partial x_i}=f(t)$$

And consequently $\sum_1^n u_i^2-\alpha\, p_i=g(t)$ because $\operatorname{div} u=0$.

We deduce therefore that:

$u_i^2-\alpha\, p_i=h_i(t),\ \forall i\in\{1,\dots,n\}$ because: $\forall i\in\{1,\dots,n\}, u_i^2-\alpha\, p_i=l_i(x_i,t)$, and we must have $\frac{\partial\left(u_i^2-\alpha\, p_i\right)}{\partial x_j}=0\ \forall j\in\{1,\dots,n\}$.

V. CONCLUSION

Theorem:

The Navier-stokes equation have a solution, Moreover, any solution (u, p) must check: $u_i^2-\alpha\, p_i=f_i(t)$ and $\frac{\partial u_i}{\partial t}=\beta\frac{\partial^2 u_i}{\partial x_i^2}$, $\forall i\in\{1,\dots,n\}$. where $u=\sum_1^n u_i e_i$ and $p=\sum_1^n p_i e_i$.

Conversely any pair (n, p) satisfying these conditions with $\operatorname{div} u=0$ is solution of the Navier-Stokes equation.

REMARKS

1- We note in the above equations the dependence between pressure, density, speed vector fields and viscosity

2- By dividing by α in the equation $u_i^2-\alpha\, p_i=f_i(t)$ we deduce that: $\rho u_i^2-p_i=\rho f_i(t)$ where $\frac{1}{\alpha}=\rho$ is the density of the fluid.

Let's $\rho=\frac{m}{V}$, we will have : $m u_i^2-Vp_i=mf_i(t)$.

And the equation $m u_i^2-Vp_i=mf_i(t)$ is linking energy, mass, pressure, temperature, volume and time ... This may not be surprising since a link between the Boltzmann equation and the Navier-Stokes has been established.

When $f_i(t)$ tends to 0, we will have $m u_i^2 \approx Vp_i$, there is therefore a tendency towards the law of an ideal gas, and the function $f_i(t)$ can be regarded as a turbulent function.

REFERENCES

1. Joseph Fourier, Théorie analytique de la chaleur, Firmin Didot Père et Fils (Paris-1822). Réédition Jacques Gabay, 1988 (ISBN 2-87647-046-2)
2. http://www.claymath.org/sites/default/files/navierstokes.pdf

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