Proof of the Riemann hypothesis.

Abstract: Consider $\chi_1(t) = \frac{1}{log(16)} log \frac{\lambda(t)}{q(t)}$, a smoothing of the unit step function. There are corresponding smoothings of its negation, given as $\chi_4 = \frac{log(16)}{\pi} e^{-t} \chi_1$, $\chi_5 = \frac{d}{dt}\ log\ \chi_1$. The Riemann hypothesis is deduced from the condition that these functions and also $\chi_4(t)/\chi_1(-t)$ are monatonic, and proofs of monatonicity are outlined.

1. Introduction. The proof of Riemann's hypothesis in this paper amounts to verifying the four inequalities which in Theorem 10 of [1] were proven sufficient.

2. Definitions and conventions.

To reduce notation and help visualize and remember things during the proof, we'll make superficial use of the notion of of expectations and density functions; and also we'll define two positive-valued smoothings of the familiar unit step function (the characteristic function of the positive reals). The first is

$$\chi_1(t) = \frac{1}{log(16)} log(\lambda(t)/q(t)).$$

The second is

$$\chi_2(t) = \frac{4}{\pi^2} K(\lambda(t))^2 \lambda(-t).$$

Of these, the first has more support on negative numbers.

Let's also let

$$\chi_3(t) = 1 - \chi_2(t)$$

so this is a smoothing of the characteristic function of the negative real numbers.

Another smoothing of the characteristic function of the negative real numbers is

$$\chi_4(t) = \frac{1}{\pi} e^{-t} log(\lambda(t)/q(t)).$$

The formulation and numbering of $\chi_1, \chi_2, \chi_3, \chi_4$ are almost intentionally nonsensical; we're only, in this paper, giving a boring technical demonstration of the four inequalities that we already know with virtual certainty must hold, because of the weight of experimental evidence about the positions of the known zero's of Riemann's function.

Quiet Remark added 31 March

After going for a walk in the woods, something occurred to me. Namely, I had been worried about how one would check things like positivity of σ, when there is essentially no conceptual limit of the analytic domain of the function. But then later, while outdoors, it struck me that I already have reformulated these questions as questions about ratios among the χ_i and their transforms. These four 'amost intentionally nonsensical' functions are probably going to be related to modular forms of weight two for $\Gamma(2)$, and their ratios related to rational functions of the simplest type, each with one pole and one zero on the Riemann sphere. Therefore Remark 9. of [1] will end up playing a role after all. These ratios are not going to be things we'll never understand, affected by considerations like Mertens' seemingly endless complexities, but just coming from ordered pairs of distinct points in the Riemann sphere after all. I should at the same time confess that in 'Nine notes on modular forms,' where I had formulas

$$\theta(0,\tau)^4 d\tau = \omega_0 = \frac{u_1}{u_1 - u_0} d\frac{u_0}{u_1}$$
$$\theta(1/2,\tau)^4 d\tau = \omega_1 = \frac{u_1}{u_0 - u_1} d\frac{u_1}{u_0}$$

satisfying the almost magical relation

$$[\omega_0 : \omega_1] = [u_0 : u_1],$$

it would have been nicer to notice that subtracting u_1 from the numerator on the right in the first formula has no effect, and subtracting u_0 from the numerator on the right in the second has no effect (because the derivative of 1 is zero), and these could have been written

$$\theta(0,\tau)^4 d\tau = \omega_0 = -d\ log\ \frac{u_0 - u_1}{u_1}$$

$$\theta(1/2,\tau)^4 d\tau = \omega_1 = -d\ log\ \frac{u_1 - u_0}{u_0}.$$

$$-d\ log(1 - (\frac{\theta(1/2,\tau)}{\theta(0,\tau)})^4)$$
$$= -d\ log\lambda,$$

and similar for the other equation.

Our 'magic ratio' only determines things up to multiplying u_0, u_1 by the same scale constant. The constant implicit in the choice of definition of λ makes

$$d\ log\lambda = i\pi\theta(1/2,\tau)^4 d\tau.$$

Then also

$$i\pi\theta(0,\tau)^4 d\tau = d\ log(1 - \frac{\theta(0,\tau)^4}{\theta(1/2,\tau)^4})$$

so

$$d\ log(1 - \frac{1}{1-\frac{1}{\lambda}}) = i\pi\theta(0,\tau)^4 d\tau.$$

The left side is $d\ log(\frac{\lambda}{1-\lambda})$. And subtracting,

$$d\ log\ (1-\lambda) = i\pi(\theta(1/2,\tau)^4 - \theta(0,\tau)^4)d\tau$$
$$= -i\pi e^{i\pi\tau}\theta(\tau/2,\tau)d\tau.$$

We get some extra invariance, as a one-form, under $\tau \mapsto \frac{-1}{\tau}$ in the case of $d\ log\frac{\lambda}{1-\lambda}$, and it follows that the function

$$\frac{d}{dt}log\frac{\lambda}{1-\lambda} = -\pi e^t\theta(0,\tau)^4 = -\pi e^t\theta(0,ie^t)^4$$

is as we knew going to be actually as a function precisely symmetric under $t \mapsto -t$.

To summarize,

$$\begin{cases} d\ log\lambda = i\pi\theta(1/2,\tau)^4 d\tau, \\ d\ log(\frac{\lambda}{1-\lambda}) = i\pi\theta(0,\tau)^4 d, \\ d\ log\ (1-\lambda) = i\pi(\theta(1/2,\tau)^4 - \theta(0,\tau)^4) = -e^{i\pi\tau}\theta(\tau/2,\tau) \end{cases},$$

and the multiplicative relation among $\lambda,\ 1-\lambda,\ \frac{\lambda}{1-\lambda}$ is what gives the additive Jacobi relation after taking logs.

Now that we've excised the value of c from the picture, it is likely that our naive observations, from looking at graphs of ratios among the χ_i as functions of t, cannot be betrayed by any subtle behaviour anymore as had troubled Mertens. And I may have been subconsciously copying Jacobi's definition of four theta functions.

Then too in fact

$$d\ log\ (\lambda/q) = i\pi\theta(1/2,\tau)^4 d\tau + i\pi d\tau.$$

Then our calculation of the derivative with respect to t, since $\tau = ie^t$ and $\frac{d}{dt}\tau = ie^t dt$, is

$$\frac{d}{dt}log(\lambda/q) = \pi e^t(1 - \theta(1/2,\tau)^4).$$

And we always view

$$log(\lambda/q)$$

as the integral of the right side, with respect to t if we wish, as a special case of the fact that it is the path integral of $d\ log(\lambda/q)$ in the complex analytic sense.

Whenever we have written

$$\frac{4}{\pi^2}K(\lambda(t))^2\lambda(-t)$$

we could use the equation

$$\frac{4}{\pi^2}K(\lambda(t))^2\lambda(-t) = \theta(1/2,\tau)^4.$$

So that our definition of χ_2 could have been given

$$\chi_2(t) = \theta(1/2,\tau)^4.$$

It is not only *subconsciously motivated* by one of Jacobi's functions, it *is* one of Jacobi's functions. The one which he calls θ_4^4.

Combining ideas a bit, we have

$$d\chi_1 = \frac{1}{log(16)} d\ log\ \lambda/q$$

$$= \frac{-i\pi}{log(16)}(1-\theta(1/2,\tau)^4)d\tau$$

$$= \frac{-i\pi}{log(16)}(1-\chi_2)d\tau$$

$$= \frac{-i\pi}{log(16)}\chi_3 d\tau.$$

Thus we may calculate χ_1 by integrating $\frac{-i\pi}{log(16)}\chi_3 d\tau$ along paths, which we may take to be paths in the triply punctured Riemann sphere.

This also gives (as we knew otherwise already)

$$d\chi_1 = \frac{\pi}{log(16)}e^t\chi_3 dt.$$

Substituting $\frac{log(16)}{\pi}e^{-t}\frac{d}{dt}\chi_1(t)$ for $\chi_3(t)$ in the formula for σ gives

$$\sigma(r,v) = \frac{log(16)}{\pi}(e^{-r/2}\chi_1(v+r/2)e^{-v+r/2}\chi_1'(v-r/2) - e^{r/2}\chi_1(v-r/2)e^{-v-r/2}\chi_1'(v+r/2))$$

where $\chi_1' = \frac{d}{dt}\chi_1$, and so

$$\sigma(r,v) = \frac{log(16)}{\pi}e^{-v}(\chi_1'(v-r/2)\chi_1(v+r/2) - \chi_1'(v+r/2)\chi_1(v-r/2))$$

$$= \frac{log(16)}{\pi} e^{-v} \chi_1(v+r/2)\chi_1(v-r/2)\frac{d}{dt} log\chi_1(v-r/2)$$
$$-\frac{log(16)}{\pi} e^{-v} \chi_1(v+r/2)\chi_1(v-r/2)\frac{d}{dt} log\chi_1(v+r/2).$$

If we define yet another function

$$\chi_5(t) = \frac{d}{dt}\ log\ \chi_1(t),$$

which at firsts glance appears to be a smoothing of the characteristic function of the negative reals, then actually the positivity of σ is equivalent to χ_5 being monatonically decreasing.

Actually also, regarding the antisymmetrization of of σ when both v and r are greater than zero, we have

$$\sigma(v,r) - \sigma(-v,r) = \frac{log(16)}{\pi}(e^{-v}\frac{d}{dt}log(\chi_1(v-r/2)) - e^{-v}\frac{d}{dt}log(\chi_1(v+r/2))$$
$$-e^{v}\frac{d}{dt}log\chi_)1(-v-r/2)) + e^{v}\frac{d}{dt}log(\chi_1(-v+r/2)).$$

By a re-arrangement of terms, this equals $\frac{log(16)}{\pi}$ times the alternating series whose terms are the product of two sequences

$$(-e^{-v}, e^{-v}, -e^{v}, e^{v})$$

$$(\chi_5(v+r/2), \chi_5(v-r/2), \chi_5(-v+r/2), \chi_5(-v-r/2)).$$

If $v > r/2$ then the second is strictly decreasing while the first decreasing, and the theorem of alternating series implies that the sum is negative. Otherwise, the argument is easier, as one has a sequence of two negative terms of large magnitude followed by two positive terms of small magnitude.

Thus

Lemma. The function σ is positive valued and its anti-symmetrization with respect to v is negative valued for positive v provided the smoothing χ_5 really is monatonic.

Here are main observations that we'll make in a rough form, we'll have to refine them when we need them. Firstly, just because χ_1 and χ_2 seem to be monatonically increasing, we have for each fixed $r > 0$ that

$$\chi_1(t + r) > \chi_1(t),$$
$$\chi_2(t + r) > \chi_2(t).$$

It follows that

$$\chi_1(t - r/2)\chi_3(t + r/2) < \chi_1(t + r/2)\chi_3(t - r/2) \quad (1)$$

and indeed the first approximates the characteristic function of the empty set, while the second approximates the characteristic function of the interval $[-r/2,\ r/2]$.

Secondly, considering the mean value of the t using density function

$$e^{(c-1)t}\chi_1(t)$$

(if it were re-scaled so that it integrates to one so that we may legally call it a density function), I claim that it is negative, meaning

$$\int_{-\infty}^{\infty} te^{(c-1)t}\chi_1(t)dt < 0 \quad (2)$$

for any choice of c in our range $0 < c < 1/2$.

This is just a calculation, the value monatonically increasing and being negative when $c = 1/2$.

3. Proof of the first inequality

The first inequality in Theorem 10, which is to be proven for all $r \geq 0$, is true because all the factors on the left side are positive. In particular, $log(\lambda(t)/q(t))$ is always greater than zero since $\lambda(t)/q(t)$ ranges (monatonically) from 1 to 16 as t ranges from $-\infty$ to ∞.

4. Proof that the second inequality follows from negativity of η.

The second inequality, which is to hold for all $r \geq 0$, simplifies to

$$log(16)^2 \int_{-\infty}^{\infty} e^{(c-1)v}\chi_1(v)((2v + r)e^{(c-1)(v+r)}\chi_1(v + r)$$

$$+(2v-r)e^{(c-1)(v-r)}\chi_1(v-r))dv<0.$$

By replacing v with $v-r/2$ and $v+r/2$ the terms can be combined to give equivalently

$$4\ log(16)^2\int_{-\infty}^{\infty} ve^{(c-1)(v-r)}\chi_1(v-r/2)\ e^{(c-1)(v+r)}\chi_1(v+r/2)dv.$$

This has the same sign as the expected value of v for a product of two symmetrically displaced translates of the same probability density function which we considered in (2). We calculated the expected value of v under one copy; it is negative. We need to be more detailed now. As an upper bound, we may take $c=1/2$, and we are looking at the expected value of v for the density function which is a produt of two symmetrically displaced translates of, let us call it

$$\phi(t)=e^{-t/2}log(\lambda(t)/q(t)).$$

The antisymmetrization of ϕ

$$\phi(t)-\phi(-t)$$

is strictly negative for $t>0$, upon multiplying by $e^{-t/2}$ this is the same statement that $\chi_4(-t)<\chi_1(t)$. We will be done if we can show the more general fact that for all $v,r>0$ the quantity

$$\begin{aligned}\phi(v+r/2)\phi(v-r/2)-\phi(-v+r/2)\phi(-v-r/2)\\ =e^t(\chi_4(v+r/2)\chi_1(v-r/2)-\chi_4(-v+r/2)\chi_1(-v-r/2))\end{aligned}$$

is strictly negative.

We might write the second factor as

$$\eta(r,v)=\chi_4(v+r/2)\chi_1(v-r/2)-\chi_4(-v+r/2)\chi_1(-v-r/2).$$

We've shown that the second inequality holds for all $r\geq 0$ if $\eta(r,v)$ is negative valued for positive v.

5. Proof that third inequality follows from positivity of σ.

The third inequality, which is to hold for $r>0$, simplifies to

$$log(16)\int_{-\infty}^{\infty} e^{(c-1)v}\chi_1(v)(log(16)(c-1)e^{(c-1)(v+r)}\chi_1(v+r)+log(16)(1-c)e^{(c-1)(v-r)}\chi_1(v-r))$$

$$+\pi e^{c(v+r)}(1-\chi_2(v+r)) - \pi e^{c(v-r)}(1-\chi_2(v-r)))dv < 0.$$

The contribution from the first two terms can be written as a sum of two integrals

$$(c-1)log(16)^2 \int_{-\infty}^{\infty} e^{(c-1)(2v+r)}\chi_1(v)\chi_1(v+r)dr$$

$$+(1-c)log(16)^2 \int_{-\infty}^{\infty} e^{(c-1)(2v-r)}\chi_1(v)\chi_1(v-r)dr.$$

Replacing v by $(v-r/2)$ in the first gives

$$(c-1)log(16)^2 \int_{-\infty}^{\infty} e^{2(c-1)v}\chi(v-r/2)\chi(v+r/2)dr$$

which is the negative of the result of replacing v by $v+r/2$ in the second, and interchanging the two χ factors.

Therefore two terms can be removed from our inequality, it is just equivalent to

$$\pi log(16) \int_{-\infty}^{\infty} e^{(2c-1)v}\chi_1(v)(e^{cr}\chi_3(v+r) - e^{-cr}\chi_3(v-r))dv < 0.$$

The first term (the positive one) would be zero if we replaced χ_1 and χ_3 with the actual characteristic functions of the positive and negative numbers respectively, because the product $\chi_1(v)\chi_3(v+r)$ would be the characteristic function of the set of numbers v which are positive but less than $-r$. Let's try to make this rigorous. Let's replace v by $v-r/2$ in the first term and by $v+r/2$ in the second. Occurrences of c in the exponents cancel, and we get

$$4\pi log(16) \int_{-\infty}^{\infty} e^{(2c-1)v}(e^{r/2}\chi_1(v-r/2)\chi_3(v+r/2) - e^{-r/2}\chi_1(v+r/2)\chi_3(v-r/2))dv < 0.$$

Let

$$\sigma(r,v) = e^{-r/2}\chi_1(v+r/2)\chi_3(v-r/2) - e^{r/2}\chi_1(v-r/2)\chi_3(v+r/2)$$

It certainly suffices if σ is (strictly) positive valued. Note that c is no longer involved in the condition.

The intuition for why this should be likely is that the second term in σ is our approximation of the empty interval of numbers larger than $r/2$ and less than $-r/2$ while the first term is the reverse.

We have proven the third inequality subject to this very reasonable conjecture about the positivity of $\sigma(r, v)$.

6. Proof that the fourth inequality follows from negativity of the anti-symmetrization of σ.

The last inequality, which is to hold for $r > 0$, simplifies to

$$log(16)\int_{-\infty}^{\infty} e^{(c-1)v}\chi_1(v)(log(16)((1+(c-1)(2v+r))e^{(c-1)(v+r)}\chi_1(v+r)$$

$$+log(16)((-1+(1-c)(2v-r))e^{(c-1)(v-r)}\chi_1(v-r)$$

$$+\pi(2v+r)e^{c(v+r)}(1-\chi_2(v+r))$$

$$-\pi(2v-r)e^{c(v-r)}(1-\chi_2(v-r)))dv > 0.$$

We can try the trick of combining the first two terms. These contribute

$$log(16)^2\int_{-\infty}^{\infty}(1+(c-1)(2v+r))e^{(c-1)(2v+r)}\chi(v)\chi(v+r)dv$$

$$+log(16)^2\int_{-\infty}^{\infty}(-1+(1-c)(2v-r))e^{(c-1)(2v-r)}\chi(v)\chi(v-r)dv.$$

Let's try the same substitution, replacing v by $v - r/2$ in the first integral. This gives

$$log(16)\int_{-\infty}^{\infty}(1+(c-1)(2v))e^{(c-1)(2v)}\chi(v-r/2)\chi(v+r/2)dv$$

Whereas replacing v by $v + r/2$ in the second one gives

$$log(16)\int_{-\infty}^{\infty}(-1+(1-c)(2v))e^{(c-1)(2v)}\chi(v+r/2)\chi(v-r/2)dv.$$

Again these cancel, again we can remove two terms, and our inequality is equivalent to

$$\pi log(16) \int_{-\infty}^{\infty} \chi_1(v) e^{(2c-1)v}((2v+r)e^{cr}\chi_3(v+r)$$

$$-(2v-r)e^{-cr}\chi_3(v-r))dv > 0.$$

Let's again try replacing v by $v-r/2$ in the first term and by $v+r/2$ in the second. Now the coefficients become $2v$ and add to $2v+2v$ and we get the equivalent inequality

$$4\pi log(16) \int_{-\infty}^{\infty} -ve^{(2c-1)v}\sigma(r,v)dv > 0.$$

Again the negative of σ occurs, so our last inequality is true if the expected value of $ve^{(2c-1)v}$ is negative for the density function underlying $\sigma(r,v)$, for each $v > 0$ and each $c \in (0,1/2)$.

Since c is less than $1/2$, the coefficient $2c-1$ of v in the exponent is negative. Thus we can again state a condition that does not depend on the particular vale of c. It suffices, for the final inequality to be true, that the antisymmetrization $\sigma(r,v) - \sigma(r,-v)$ takes negative values for $r, v > 0$.

7. Statement of theorem.

Theorem. The Riemann hypothesis is true if for all $v, r > 0$ the quantity

$$\eta(r,v) = \chi_4(v+r/2)\chi_1(v-r/2) - \chi_4(-v+r/2)\chi_1(-v-r/2)$$

is less than zero, while for all real v and all $r > 0$ the quantity

$$\sigma(r,v) = e^{-r/2}\chi_1(v+r/2)\chi_3(v-r/2) - e^{r/2}\chi_1(v-r/2)\chi_3(v+r/2)$$

is greater than zero, while also for all $r, v > 0$ the anti-symmetrization $\sigma(r,v) - \sigma(r,-v)$ is less than zero. If χ_5 is monatonic then σ is indeed positive valued and its anti-symmetrization in the second coordinate is indeed negative valued for positive values of that coordinate. If $\chi_4(t)/\chi_1(-t)$ is monatonic then $\eta(r,v)$ is indeed greater than zero for all $r, v > 0$.

Therefore the Riemann hypothesis is a consequence of monatonicity of $\chi_5(t)$ and $\chi_4(t)/\chi_1(-t)$.

Proof. In earlier sections we proved that the four inequalities of Theorem 10 of [1] follow from negativity of η, positivity of σ and negativity of the antisymmetrization of σ. In a lemma in a remark we showed that the last two conditions would be proven once it is established that χ_5 is monatonic. The remaining assertion is this lemma

Lemma. η is negative valued for positive v if the ratio $\chi_4(t)/\chi_1(-t)$ is monatonic.

Proof. The negativity of η is equivalent to the equation

$$\frac{\chi_4(v+r/2)\chi_1(v-r/2)}{\chi_1(-v-r/2)\chi_4(-v+r/2)} < 1.$$

which is equivalent to

$$\frac{\chi_4(v+r/2)}{\chi_1(-v-r/2)} < \frac{\chi_4(-v+r/2)}{\chi_4(v-r/2)}$$

the condition that $\chi(t)_4(t)/\chi_1(-t)$ is monatonically decreasing.

8. Concluding remarks.

Remark. I believe that these conditions on σ are true, but even if not, they can be weakened while still implying Riemann's hypothesis.

Remark. Let's prove monatonicity of χ_1.

We have $\chi_1 = \frac{1}{log(16)} log(\frac{\lambda}{q})$ so

$$\frac{d}{dt}\chi_1(t) = \frac{1}{log(16)}\frac{d}{dt}(log(\lambda) + \pi e^t)$$

$$= \frac{1}{log(16)}(-\pi\theta(1/2, ie^t)^4 + \pi e^t)$$

$$\frac{\pi e^t}{log(16)}(1 - \theta(1/2, ie^t)^4.$$

Now it is sensible to re-interpret the square of the theta function using the elliptic integral

$$\theta(1/2, ie^t)^2 = \frac{2}{\pi}(1-\lambda)\int_0^{\pi/2} \frac{d\theta}{1-\lambda sin^2\theta}$$

and for $0 < \lambda < 1$ this takes a value less than 1, so the derivative of χ_1 is always greater than zero.

Remark. Let's next outline a proof of monatonicity of $\chi_5(t) = \frac{d}{dt}log\chi_1(t) = \frac{d}{dt}log\ \frac{1}{log(16)}log\frac{\lambda}{q} = \frac{d}{dt}log\ log\frac{\lambda}{q}$. (Note the constant contributes zero.)

We write

$$\frac{d}{dt}log\ log\frac{\lambda}{q} = \frac{1}{log\ \frac{\lambda}{q}}\frac{d}{dt}log\frac{\lambda}{q}$$

$$= \frac{1}{(log(\lambda) + \pi e^t)}(\frac{d}{dt}(log(\lambda)) + \pi e^t)$$

$$= \frac{-\pi e^t\theta(1/2, ie^t)^4 + \pi e^t}{log(\lambda) + \pi e^t}$$

$$= \frac{1 - \theta(1/2, ie^t)^4}{1 + \frac{e^{-t}}{\pi}log(\lambda)}$$

(This is incidentally shows that $\chi_5(t) = \chi_3(t)/\chi_4(t)$.) For $t >> 0$, there is no difficulty. Reinterpreting the denominator as $log(\lambda/q)$ noting that as $q \to 0$ this tends to $log(16)$ which is not zero, the quantity is well approximated by πe^t times the first term in $(1 - \theta(1/2, ie^t)^4)$ which is $8e^{-\pi e^t}$. Thus our approximation for $t >> 0$ is the decreasing function

$$\frac{8\pi}{log(16)}e^{t-\pi e^t} = \frac{2\pi}{log(2)}e^{t-\pi e^t}.$$

For $t << 0$ we must think more carefully. The second term in both the numerator and denominator tend to 0 as t tends to $-\infty$. Then the derivative of the ratio is well approximated by

$$\frac{d}{dt}(-\theta(1/2, ie^t)^4 - \frac{e^{-t}}{\pi}log(\lambda))$$

By coincidence this same expression just represents the *second* derivative of

$$\frac{e^{-t}}{\pi}log(\lambda).$$

Remaining negative would be consistent with this negative-valued function, which indeed we know does remain negative-valued, having a negative second derivative due to approaching a horizontal asymptote as t tends to $-\infty$.

What we should do is use the identity, writing

$$\lambda = \lambda(e^{-\pi e^t})$$

that also

$$\lambda = 1 - \lambda(e^{-\pi e^{-t}}).$$

Then we can estimate the second derivative of

$$\frac{e^{-t}}{\pi}log(\lambda)$$

$$= \frac{e^{-t}}{\pi}log(1 - 16e^{-\pi e^{-t}} + 128e^{-2\pi e^{-t}} - 704e^{-3\pi e^{-t}}...)$$

and if we just for ony look at the first term

$$-\frac{16}{\pi}e^{-t-\pi e^{-t}}$$

the first derivative of this is

$$\frac{16}{\pi}(-\pi e^{-t} + 1)e^{-t-\pi e^{-t}}$$

and the second derivative of this one term is

$$\frac{16}{\pi}(-\pi^2 e^{-2t} + 3e^{-t} - 1)e^{-t-\pi e^{-t}}$$

which is negative for all values of t (and should approximate our actual derivative for $t << 0$). To make this rigorous, let's go through it again. To save notation let's write

$$\alpha = \theta(1/2, ie^t)^4$$

$$\beta = \frac{e^{-t}}{\pi} log(\lambda).$$

We are trying to show that

$$\frac{1-\alpha}{1+\beta}$$

is monatonically decreasing with t. We can rewrite this as

$$1 + \frac{\beta'}{1+\beta}.$$

The derivative of this, again denoting derivatives using priming notation, is

$$\frac{\beta''(1+\beta) - (\beta')^2}{(1+\beta)^2}$$

and both terms which are squares are nonnegative.

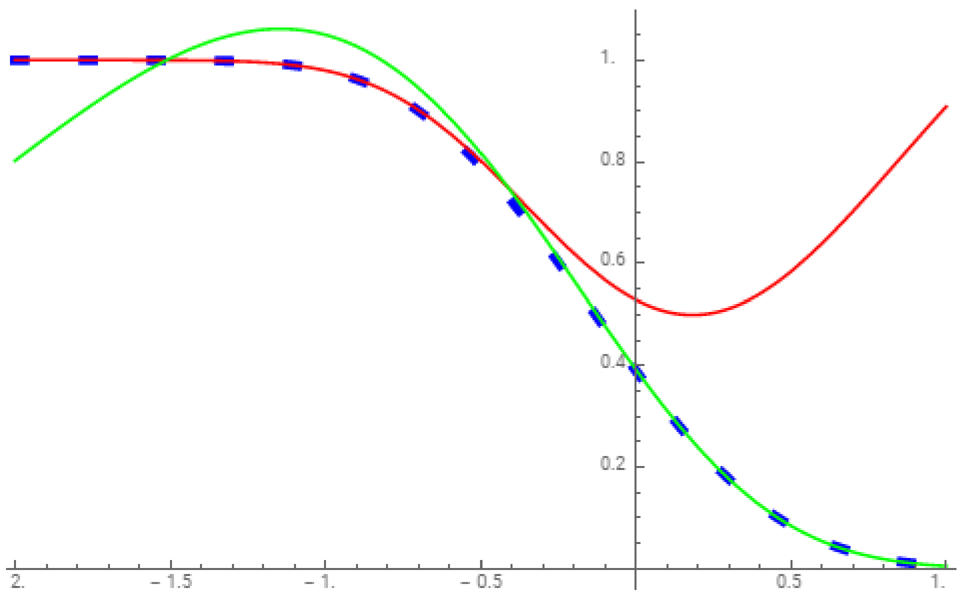
The first term in the series expression for β as a Maclaurin series with variable $e^{-t-\pi e^{-t}}$ is

$$-\frac{16}{\pi} e^{-t-\pi e^{-t}}$$

Our expressions for β' and β'' are convergent power series with the same variable, but with coefficients which are polynomials in e^{-t}. By restricting t to be less than a particular negative number, we can arrange that ignoring all but the first power of the 'Maclaurin' type variable does not affect the sign of the whole expression; this is by comparing exponential versus polynomial growth rates. Then for $t << 0$ the sign is indeed determined by the expression which we already considered.

Thus we have proven that $\chi_5(t)$ is monatonic when $t >> 0$ or when $t << 0$. We can also evaluate it in an intermediate rangee informally, but we cannot say that this proof of monatonicity of χ_5 is finished because we have not established that the intermediate range, might be denoted $-\infty >> t << \infty$, where neither separate proof applies, is fully investigated by looking at graphs. Practically speaking we know that it will be possible to fill this gap in the proof; in a graph drawn in the the ordinary way, the one-term approximation on each side looks indistinguishable from the graph of the function itself once $|t| > 1$ and in a larger range the graph of the function itself still has a noticeable downwards slope.

Here, the green graph made by Wolfram Alpha shows $\frac{8\pi}{log(16)}e^{t-\pi e^t}$ which is our monatonic approximation for $t >> 0$. The red graph shows $1+\frac{16}{\pi}(1-\pi e^{-t})e^{-t-\pi e^{-t}}$ which is our monatonic approximation for $t << 0$. The dashed blue graph describes our actual function $\chi_5(t)$ which is the same as $\chi_3(t)/\chi_4(t)$, and appears to have nonzero gradient in the range where neither approximation is reliable.



Remark. Let's prove monatonicity of χ_4. We have

$$\frac{d}{dt}\chi_4(t) = \frac{d}{dt}\frac{1}{\pi}e^{-t}log(\frac{\lambda}{q})$$

$$= \frac{e^{-t}}{\pi}(-log(\frac{\lambda}{q}) + \pi e^t(1 - \theta(1/2, ie^t)^4))$$

$$= \frac{e^{-t}}{\pi}(-log(\lambda) - \pi e^t + \pi e^t - \theta(1/2, ie^t)^4)$$

$$= \frac{e^{-t}}{\pi}(-log(\lambda) - \pi e^t\theta(1/2, ie^t)^4)$$

We need this to be negative, so we need

$$log(\lambda(t)) + \pi e^t\theta(1/2, ie^t)^4$$

to be always greater than zero, as it appears to be. Multiplying this last function by $\frac{1}{log(16)}$ it looks like yet another positive-valued smoothing of the characteristic function of the positive reals. We might write

$$\chi_6(t) = \frac{1}{log(16)}(log(\lambda(t)) + \pi e^t \theta(1/2,\tau)^4)$$

another smoothing of the unit step function, and we have that χ_4 really is monatonic if and only if χ_6 really is positive valued.

Recall that what we call $\lambda(t)$ is what elsewhere is usually denoted n $\lambda(\tau)$ for $\tau = ie^t$.

Anyway, the second term here is minus the derivative of the first, so the $\chi_6(t) = 0$ if and only if $\frac{d}{dt} log\ log \lambda(t) = 1$. But $log\ log\ \lambda$ is a composite of three monatonically decreasing functions, so there is no such value of t.

Remark. Here is the beginning of a proof of monotonicity of $\chi_4(t)/\chi_1(-t)$. This equals $\frac{log(16)}{\pi}$ times

$$e^{-t}\frac{\chi_1(t)}{\chi_1(-t)}$$

For $t >> 0$ the numerator in $\frac{\chi_1(t)}{\chi_1(-t)}$ approaches 1 rapidly and the whole expression ought to be well approximated by $e^{-t}/\chi_1(-t)$. For $t << 0$ the denominator approaches 1 rapidly and this ought to be well approximated by $e^{-t}\chi_1(t)$. Letting u be the positive number $-t$ when $t << 0$ this is the reciprocal of $e^{-u}/\chi_1(-u)$, that is to say, once the absolute value of t is large, negating t has approximately the same effect as reciprocating our quantity, and once we prove monatonicity for $t >> 0$ it should follow for $t << 0$. To show $\frac{e^{-t}}{\chi_1(-t)}$ is monatonically decreasing for $t >> 0$ is the same as showing that $e^t\chi_1(-t)$ is monatonically increasing for $t >> 0$. This is the same as showing that $\chi_1(-t) + \frac{d}{dt}\chi_1(-t)$ is positive for $t >> 0$.

We calculate $log(16)$ times this quantity, being careful to use the chain rule for $-t$, as

$$log(\lambda(-t)) - \pi e^{-t} - \pi e^{-t}(1 - \theta(1/2, ie^{-t})^4$$

$$= log(\lambda(-t)) + \pi e^{-t}\theta(1/2, ie^{-t})^4.$$

The first term is negative and the second positive, for $t >> 0$, Since $\lambda(-t)$ is near 1 we may approximate the first term very closely by $\lambda(-t) - 1$ and using the rule that $\lambda(-t) = 1 - \lambda(t)$ this is the same as $-\lambda(t)$. Thus our approximation now is

$$-\lambda(t) + \pi e^{-t}\theta(1/2, ie^{-t}).$$

The leading term of the *lambda* series is $16e^{-\pi e^t}$. For the theta series, we use

$$\theta(1/2, -1/\tau)^4 = -\tau^2(\theta(0, \tau)^4 - \theta(1/2, \tau)^4)$$

and setting $\tau = ie^t$ the leading term on the right is $16e^{-t}e^{2t}e^{-\pi e^t}$ where the 16 comes from the fact that two coefficients equal to the number of ways of writing 1 as a sum of four squares are added together. Thus our approximation for the derivative, for $t >> 0$, is

$$\frac{d}{dt}(\chi_4(-t)/\chi_1(t)) \cong 16(\frac{-e^t}{\pi} + e^{2t})e^{-\pi e^t}.$$

As in an earlier argument, this does establish monatonicity of $\chi_4(t)/\chi_1(-t)$ for $t >> 0$ and as we observed also then for $t << 0$. We can look at graphs of the function and observe monatonicity for intermediate values, but have not yet proven that the observed values cover the full necessary intermediate range.