FORECAST VOLATILITY AND VALUE AT RISK WITH A GARCH MODEL

Master's Thesis in Financial Econometrics

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Abstract

In this paper, we propose to forecast the Value at Risk of the french stock index, the CAC40, with a GARCH(1,1) model. Hence, we propose to evaluate the quality of our estimations with backtesting techniques as the Kupiec's test (1995). We find that, even though the leptokurtik distribution that assumes the returns on asset of the index, we much more tend towards to overestimate the Value at Risk.

keywords : Value at Risk, GARCH, Estimation, Backtesting.
JEL Classification : C22, C52, C53, G15.

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1 Introduction

The volatility is a central problem in financial markets. An investor, in order to make profits, has to be very careful concerning the volatility of its portfolio, and the finance is an uncertain universe, as it is shown by numbers of financial crisis that the world has been through. Hence, has emerged a important research from academic and financial institutions to cultivate tools for market risk estimations. One of the most famous risk measure is the Value-at-Risk (VaR). The VaR represents the maximal potential thaht could make an investor on the value of his security portfolio reachable with a given probability and a given time horizon (*Angelidis et al*, 2004) [1].Then, the VaR is the worse expected amount of loss for a given confidence level. This tool has been used the first time in 1980 by the american bank *Bankers Trust*. It has been generalized by *JP Morgan* in 1990 with its *riskmetrics* system. Later, the *Basel* comimitte has established that tool in the banking system in obliging bankers to calculate the VaR for their portofolios that is supposed to avert some financial follies.

The calculation of the VaR requires to estimate the volatility of the security, this is to say its variance. As a lot of economic phenomenons, financial instruments have an heteroscedastic variance. The autoregressive conditional heteroscedasticity (ARCH) model and the general autoregressive conditional heteroscedasticity (GARCH) model, developped by Engle (1982)[4] and Bollerslev (1986) [2] permit to re-evaluate the property of homoscedasticity that is usually used within the scope of the classic linear model. These models capture the fluctuations in variance over time presents in financial instruments. Since the developpement of both models ARCH-GARCH, a lot of varieties of these have appeared (*Bollerslev*, 2010) [3]. Nonetheless, there is no consensus concerning which of these models is able to realize the best volatility estimation.

In this thesis, we firstly expose some theoretical points concerning the VaR and the ARCH-GARCH models. In the second part, we implement the GARCH model in order to forecast the VaR of the french stock index, the CAC40.

2 Theory and definitions

2.1 The Value at Risk

2.1.1 Definitions

The VaR is only the fractile of the distribution of profit and loss associated to the ownership of an asset or an assets portfolio for a given period. It just represents the information included on the left of the distribution of returns. Then, if we consider a coverage ratio of $\alpha\%$, the VaR correspond to the fractile of $\alpha\%$ level of the distribution of loss and profit during the possession of an asset :

$$VaR(\alpha) = F^{-1}(\alpha)$$

where $F(.)$ refers to the distribution function associated to the distrbution of loss and profit.

The VaR depends on three points : the distribution of profits and losses of the portfolio, the level of confidence and the period of the security possession.

The chosen level of confidence is a parameter included between 0 an 1 (Usually 95% or 99%) that permits to control the probability to have a return on asset superior or equal to the VaR. For example, on the figure 1,

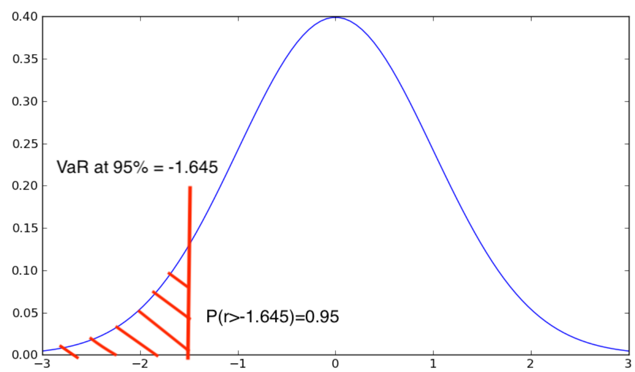


Figure 1: An example of VaR with Gaussian distribution.

one can see the distribution of negative returns are on the left and positive returns are on the right. Then, the VaR defined for a level of confidence of 95% ($\alpha = 5\%$) equal to $1,645$. Put another way, there is 95% chances for the return on the asset r to be at less equal to $-1,645$ on the period of possession.

$$P\left[r < VaR(0,05)\right] = P\left[r < -1,645\right] = 0,05$$

2.1.2 The conditional Value at Risk

One can distinguish the conditional distribution from the non-conditional distribution. Let notice R, the return on an asset. Let's suppose that the return is an real-valued random variable of profit and loss with its density $f_R(r)$, $\forall r \in \mathbb{R}$. For this random variable, we can define its conditional density to a set of information, noticed Ω. So, $f_R(r|\Omega)$, $\forall r \in \mathbb{R}$ the conditional density associated to the return of an asset.

The VaR conditional to a set of information Ω, associated to a coverage ratio α, fit it with the order's fractile α of the conditional distribution of losses and profit. It is written as :

$$VaR(\alpha) = F_R^{-1}(\alpha|\,\Omega)$$

However, in this case, we do not have considered the temporal dimension yet. Thus, let us consider R_t, the return with the temporal subscript t, and so $f_R(r|\Omega_t)$, $\forall r \in \mathbb{R}$, this distribution of profits and losses for the same date. This density can be different over time, but we generally consider it as invariable in time. This comes down to consider the returns as identically distributed. This is particularly this hypothesis that permits to forecast VaR in the case of parametric models as the GARCH model.

As a consequence, the VaR at the time t, computed conditionally to a set of information Ω_t is noticed as follows :

$$VaR_t(\alpha) = F_{R_t}^{-1}(\alpha|\,\Omega)$$

2.2 The ARCH Model

The return R_t of a security is defined as :

$$R_t = 100 \log\left(\frac{P_t}{P_{t-1}}\right) \tag{1}$$

where P_t is the closing price of the security day t. The return consist in two parts, a predictable and an unpredictable part :

$$R_t = E\left(R_t|I_{t-1}\right) + \epsilon_t \tag{2}$$

where I_{t-1} is all available information before $t-1$. ϵ_t represents the residuals, this is to say the unpredictable return. The conditionnal return assumes an autoregressive process :

$$E\left(R_t|I_{t-1}\right) = \alpha_0 + \sum_{i=1}^{q} \alpha_i R_{t-i} \tag{3}$$

The unpredictable part of the returns can be expressed as :

$$\epsilon_t = z_t\sqrt{h_t} \tag{4}$$

where z_t indicates a weak white noise such that $E\left(z_t\right) = 0$ and $E\left(z_t^2\right) = \sigma_z^2$.

The ARCH(q) model is defined as :

$$V(\epsilon_t|\epsilon_{t-1}) = h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 \tag{5}$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$ for $i = 1, ..., p$. Finally, the ARCH presents a process today's variance depends on its own previous variance. This can permit us to capture the volatilty of financial instruments.

Let's consider an ARCH(1) :

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 \tag{6}$$

With transformation :

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 \Leftrightarrow \epsilon_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + (\epsilon_t^2 - h_t) \tag{7}$$

It leads to :

$$\epsilon_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + u_t \tag{8}$$

where $u_t = (\epsilon_t^2 - h_t)$ assumes the property of innovation process because $E\left(u_t|\epsilon_{t-1}\right) = 0$.

By recurrence, we find its variance $E\left(\epsilon_t^2\right)$ that is steady in time, under stationnarity conditions :

$$\begin{aligned} & E\left(\epsilon_t^2\right) = \alpha_0 + \alpha_1 E\left(\epsilon_{t-1}^2\right) \\ \Leftrightarrow & E\left(\epsilon_t^2\right) = \alpha_0 + \alpha_1\alpha_0 + \alpha_1^2 E\left(\epsilon_{t-2}^2\right) \\ \Leftrightarrow & E\left(\epsilon_t^2\right) = \alpha_0 \left(1 + \alpha_1 + \alpha_1^2 + ... + \alpha_1^{h-1}\right) + \underbrace{\alpha_1^h E\left(\epsilon_{t-h}^2\right)}_{\Longrightarrow 0} \\ \Longrightarrow \lim_{h \to +\infty} & E\left(\epsilon_t^2\right) = \alpha_0 \sum_{h=0}^{+\infty} \alpha_1^h = \frac{\alpha_0}{1 - \alpha_1} \end{aligned}$$

for $\alpha_1 < 1$ and $\forall t, \forall h, E\left(\epsilon_{t-h}^2\right) < \infty$.

The ARCH(1) model gives us the forecast for next period :

$$\hat{h}_{t+1} = \hat{\alpha_0} + \hat{\alpha_1} \epsilon_t^2 \tag{9}$$

2.3 The GARCH model

The GARCH(p,q) is defined as follows :

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_j h_{t-i} \tag{10}$$

With its residual term :

$$\epsilon_t = z_t \sqrt{h_t} \tag{11}$$

where z_t is a white noise.

The GARCH model assumes the following conditional moments :

$$E\left(\epsilon_t|\epsilon_{t-1}\right) = 0 \tag{12}$$

$$V\left(\epsilon_t|\epsilon_{t-1}\right) = h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i} \tag{13}$$

As the ARCH model, the ϵ_t^2 process can be represented as an innovation process μ_t :

$$\mu_t = \epsilon_t^2 - h_t \tag{14}$$

By substituting (14) in (13), we have :

$$\epsilon_t^2 - \mu_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \left(\epsilon_{t-1}^2 - \mu_{t-i}\right)$$
$$\Leftrightarrow \epsilon_t^2 = \alpha_0 + \sum_{i=1}^{max(p,q)} \left(\alpha_i + \beta_i\right) \epsilon_{t-i}^2 + \mu_t - \sum_{i=1}^{p} \beta_i \mu_{t-i}$$

Finally, the ϵ_t^2 of a GARCH(p,q) representation can be expressed by an ARMA[max(p,q), q].

Let's consider a GARCH(1,1) :

$$\epsilon_t = z_t \sqrt{h_t}$$

$$V(\epsilon_t|\epsilon_{t-1}) = h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 h_{t-1} \tag{15}$$

That can be represented as follows :

$$\epsilon_t^2 = \alpha_0 + \left(\alpha_1 + \beta_1\right) \epsilon_{t-1}^2 + \mu_t - \beta_1 \mu_{t-1} \tag{16}$$

where $\mu_t = \epsilon_t^2 - h_t$ is an innovation process for ϵ_t^2.

By recurrence, we deduce from this its variance $E\left(\epsilon_t^2\right)$ that also is steady in time :

$$E\left(\epsilon_t^2\right) = \alpha_0 + \left(\alpha_1 + \beta_1\right) E\left(\epsilon_{t-1}^2\right) + E\left(\underbrace{\mu_t - \beta_1 \mu_{t-1}}_{=0}\right)$$
$$\Leftrightarrow E\left(\epsilon_t^2\right) = \alpha_0 + \left(\alpha_1 + \beta_1\right) E\left[\alpha_0 + \left(\alpha_1 + \beta_1\right) E\left(\epsilon_{t-2}^2\right) + E\left(\underbrace{\mu_{t-1} - \beta_1 \mu_{t-2}}_{=0}\right)\right]$$
$$\Leftrightarrow E\left(\epsilon_t^2\right) = \alpha_0 + \alpha_0 \left(\alpha_1 + \beta_1\right) + \left(\alpha_1 + \beta_1\right)^2 + \left(\alpha_1 + \beta_1\right) \epsilon_{t-2}^2$$
$$\Leftrightarrow E\left(\epsilon_t^2\right) = \alpha_0 \left[1 + \left(\alpha_1 + \beta_1\right) + ... + \left(\alpha_1 + \beta_1\right)^{h-1}\right] + \left(\alpha_1 + \beta_1\right)^h \underbrace{E\left(\epsilon_{t-h}^2\right)}_{\Longrightarrow 0}$$

$$\Longrightarrow \lim_{h \to +\infty} E\left(\epsilon_t^2\right) = \alpha_0 \sum_{h=0}^{+\infty} \left(\alpha_1 + \beta_1\right)^h = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$$

for $\alpha_1 + \beta_1 < 1$ and $\forall t, \forall h, E\left(\epsilon_{t-h}^2\right) < \infty$.

The GARCH(1,1) model gives us the forecast for next period :

$$\hat{h}_{t+1} = \hat{\alpha_0} + \hat{\alpha_1} \epsilon_t^2 + \hat{\beta}_1 h_{t-1} \tag{17}$$

2.4 Estimation

The estimation of parameters can be made by the maximum likelihood or the quasi-maximum likelihood method. For both cases, we need to assume the Gaussian hypothesis of residuals.

By taking the representation of Gouriéroux (1992)[5], we consider the following conditionnal moments :

$$E\left(Y_t|Y_{t-1}, X_t\right) = m_t\left(Y_{t-1}, X_t, \theta\right) = m_t \theta$$
$$V(Y_t|Y_{t-1}, X_t) = h_t\left(Y_{t-1}, X_t, \theta\right) = h_t \theta$$

where θ represents the parameters.

Then, the likelihood function associated to a sample of T observations $(y_1, y_2, ..., y_T)$ of Y_t under the Gaussian hypothesis of the conditionnal law of Y_t knowing Y_{t-1} and X_t is written :

$$\mathcal{L}_T(\theta) = \prod_{t=1}^{T} \frac{1}{\sqrt{h_t(\theta)2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y_t - m_t(\theta)}{h_t(\theta)}\right)\right]^2 \tag{18}$$

From there, we deduce the log-likelihood :

$$\log \mathcal{L}_T(\theta) = -\frac{1}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T} \log h_t(\theta) - \frac{1}{2}\sum_{t=1}^{T}\left[\frac{y_t - m_t(\theta)}{h_t(\theta)}\right]^2 \tag{19}$$

If we consider the case of a linear regression $Y_t = X_t\beta + \epsilon_t$ with ARCH(q) errors $\epsilon_t = z_t\sqrt{h_t(\theta)}$, $z_t \sim \mathcal{N}(0,\, 1)$, with its moments $E\left(\epsilon_t|\epsilon_{t-1}\right) = 0$ and $V(\epsilon_t|\epsilon_{t-1}) = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2$, we have :

$$E(Y_t|Y_{t-1}, X_t) = m_t\theta = X_t\beta \tag{20}$$

$$V(Y_t|Y_{t-1}, X_t) = h_t(\theta) = \alpha_0 + \sum_{i=1}^{q} \alpha_i (Y_{t-1} - \beta X_{t-i})^2 \tag{21}$$

where $\theta = (\beta, \alpha_0, \alpha_1, \ldots, \alpha_q) \in \mathbb{R}^{q+2}$.

Thus in this particular case, the log-likelihood is written :

$$\begin{aligned} \log \mathcal{L}_T(\theta) = &-\frac{T}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T} \log\left[\alpha_0 + \sum_{i=1}^{q} \alpha_i (Y_{t-i} - \beta X_{t-i})^2\right] \\ &- \frac{1}{2}(y_t - X_t\beta)^2 \left[\alpha_0 + \sum_{i=1}^{q} \alpha_i (Y_{t-i} - \beta X_{t-i})^2\right]^{-1} \end{aligned}$$

Estimators of the likelihood maximum or the quasi-likelihood maximum under the Gaussian hypothesis, represented $\hat{\theta}$, of parameters $\theta \in \mathbb{R}^K$, satisfy a non-linear system of K equations :

$$\left.\frac{\partial \log \mathcal{L}_T(\theta)}{\partial \theta}\right|_{\theta=\hat{\theta}} \tag{22}$$

Then, we have in the generalized model :

$$\begin{aligned} \left.\frac{\partial \log \mathcal{L}_T(\theta)}{\partial \theta}\right|_{\theta=\hat{\theta}} = &-\frac{1}{2}\sum_{t=1}^{T}\left[\frac{1}{h_t(\hat{\theta})} + \frac{[y_t - m_t(\hat{\theta})]^2}{h_t(\hat{\theta})}\right] \left.\frac{\partial h_t(\theta)}{\partial \theta}\right|_{\theta=\hat{\theta}} \\ &+ \sum_{t=1}^{T}\left[\frac{y_t - m_t(\hat{\theta})}{h_t(\hat{\theta})}\right] \left.\frac{\partial m_t(\theta)}{\partial \theta}\right|_{\theta=\hat{\theta}} \end{aligned}$$

The system of likelihood equations can be decomposed in two simple equations when $\theta = (\alpha\,\beta)'$, where α only appears in the conditional average and β only in conditional variance as follows :

$$\left.\frac{\partial \log \mathcal{L}_T(\theta)}{\partial \alpha}\right|_{\theta=\hat{\theta}} = \sum_{t=1}^{T} \left[\frac{y_t - m_t(\hat{\alpha})}{h_t(\hat{\beta})}\right] \left.\frac{\partial m_t(\alpha)}{\partial \alpha}\right|_{\theta=\hat{\theta}} \tag{23}$$

$$\left.\frac{\partial \log \mathcal{L}_T(\beta)}{\partial \beta}\right|_{\theta=\hat{\theta}} = -\frac{1}{2} \sum_{t=1}^{T} \left[\frac{1}{h_t(\hat{\beta})} + \frac{[y_t - m_t(\hat{\alpha})]^2}{h_t(\hat{\beta})^2}\right] \left.\frac{\partial h_t(\theta)}{\partial \beta}\right|_{\theta=\hat{\theta}} \tag{24}$$

In the general case of the quasi-maximum likelihood, its estimator is asymptotically Gaussian :

$$\sqrt{T}(\hat{\theta} - \theta) \underset{T\to+\infty}{\longrightarrow} \mathcal{N}(0,\, J^{-1} I J^{-1}) \tag{25}$$

With its asymptotic variance-covariance matrix of the quasi-likelihood maximum :

$$J = E_0\left[-\frac{\partial^2 \log \mathcal{L}_T(\theta)}{\partial\theta\partial\theta'}\right] \qquad I = E_0\left[\frac{\partial \log \mathcal{L}_T(\theta)}{\partial\theta}\frac{\partial \log \mathcal{L}_T(\theta)}{\partial\theta'}\right] \tag{26}$$

where E_0 represents the expectation took in relation to the real law.

But in practise, matrixes J and I are directly estimated as it follows :

$$\hat{I} = \frac{1}{T}\sum_{t=1}^{T} \left.\frac{\partial \log \mathcal{L}_T(\theta)}{\partial\theta}\right|_{\theta=\hat{\theta}} \left.\frac{\partial \log \mathcal{L}_T(\theta)}{\partial\theta'}\right|_{\theta=\hat{\theta}} \tag{27}$$

$$\hat{J} = \frac{1}{T}\sum_{t=1}^{T} \left.\frac{\partial^2 \log \mathcal{L}_T(\theta)}{\partial\theta}\right|_{\theta=\hat{\theta}} \tag{28}$$

and the estimated variance of the estimator $\hat{\theta}$ verifies the following condition :

$$V\left[\sqrt{T}(\hat{\theta} - \theta)\right] = \hat{J}^{-1}\hat{I}\hat{J}^{-1} \tag{29}$$

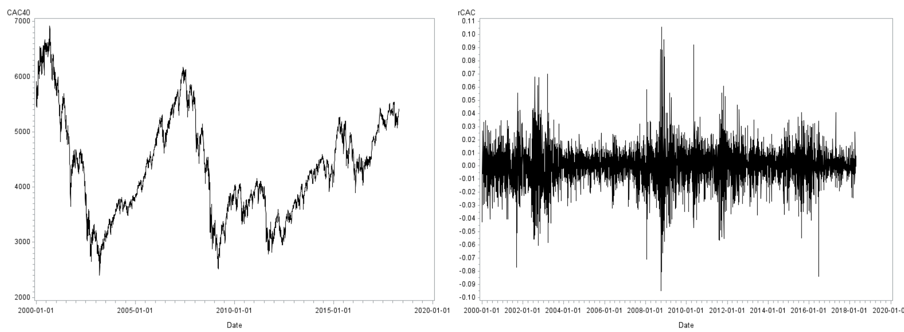
In this case of maximum likelihood, the real distribution assumes a Gaussian law, the asymptotic variance-covariance matrix is written as :

$$V\left[\sqrt{T}(\hat{\theta} - \theta)\right] = \hat{J}^{-1} \tag{30}$$

3 Application

3.1 Forecast strategy

In this article, we focus our analysis on the french CAC40 Index starting from 2000 to 2018. The VaR forecast for a given level of confidence of



(a) The CAC40 Index.

(b) The CAC40's log returns.

Figure 2: The CAC40

$1-\alpha\%$ at the the time $t+1$ simply corresponds to the fractile at the α level of the conditional distribution of profits and losses. Then we formally notice the forecast $VaR_{t+1|t}(\alpha)$:

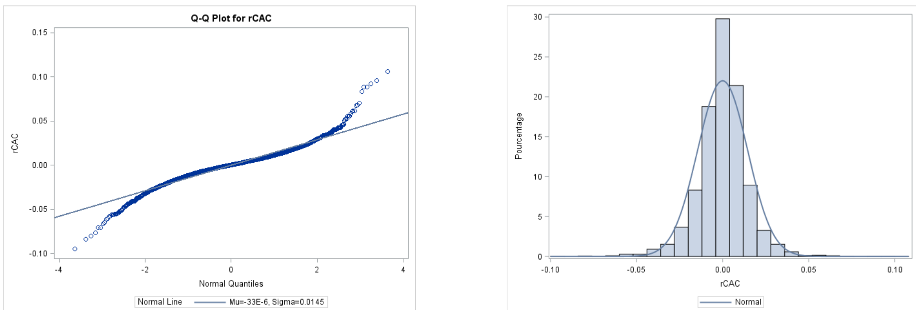
$$VaR_{t+1|t}(\alpha) = F^{-1}_{R_{t+1}}(\alpha|\,\Omega_t)$$

where $F^{-1}_{R_{t+1}}(\alpha|\Omega_t)$ corresponds to the distribution function associated to returns distribution function at the time $t+1$, conditionally to a set of information Ω_t available at the time t.

The strategy, in order to forecast the VaR with the GARCH model, consist in two steps. In the first time, we need to make an hypothesis concerning the condtional distribution of returns on asset, then estimate parameters of the GARCH model overtime with the likelihood maximum method. In the second time, we deduce from the estimated GARCH model a forecast of conditional variance that, always in considering the hypothesis of the return distribution, permit to forecast the fractile of the distribution of profits and losses at $t+1$.

Mean	-0.000032	**Median**	0.00030
Std deviation	0.0144947	**Variance**	0.000210
Skewness	-0.039626	**Kurtosis**	5.0407267
Min	-0.09471537	**Max**	0.10594589

Table 1: Descriptive statistics.



(a) QQ-plot. (b) Normality.

Figure 3: Test for Normality.

According to the table 1, we can observe that the mean is negative. It results that the returns on the french index are on average negatives. Also, the distribution assumes a negative skewness. Then, the statistical distibution of returns is shift towards the right of the median that indicates a tail-end distribution spread towards the left. So there is more negative returns than positive returns. Furthermore, the kurtosis is positive. The returns of the french index assumes a leptokurtic distribution. So it means that tail-end distribution is more bushy than the normal distribution. Finally, we see that this distribution does not totally assume the normality hypothesis.

Let us consider our model written as follows :

$$r_t = c + \epsilon_t$$

$$\epsilon_t = z_t\sqrt{h_t} \quad z_t \sim \mathcal{N}(0,\, 1)$$
$$h_t = \alpha_0 + \alpha_1\epsilon_{t-1}^2 + \beta_1 h_{t-1}$$

where z_t is a homoscedastic white noise, and parameters α_0, α_1, β_1, v and c verify the next constraints : $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and $v > 2$. $h_t = E(\epsilon_t|\epsilon_{t-1})$ refers to the conditional variance of the residual term ϵ_t, so of the returns r_t.

So the converging estimators $\hat{\alpha_0}$, $\hat{\alpha_1}$, $\hat{\beta_1}$, $\hat{\beta_1}$, $\hat{v}$ and $\hat{c}$ (obtained by maximum likelihood or the quasi-maximum likelihood methods) of corresponding parameters. From this, we forecast the conditional variance of returns at $t+1$ as it follows :

$$\hat{h}_{t+1} = \hat{\alpha}_0 + \hat{\epsilon}_t^2 + \beta_1 \hat{h}_t \quad (31)$$

with a given h_1.

Let us notice the $VaR_{t+1|t}(\alpha)$ the forecast of the VaR at a $1-\alpha$ level predicted at $t+1$ conditionnaly to the information available at t. Then it is formally written :

$$P\left[r_{t+1} < VaR_{t+1|t}(\alpha) \middle| \Omega_t\right] = \alpha \quad (32)$$

$$\Leftrightarrow P\left[z_{t+1} < \frac{VaR_{t+1|t}(\alpha) - c}{\sqrt{h_{t+1}}} \middle| \Omega_t\right] = \alpha \quad (33)$$

We deduce from (34) the forecasted value of the VaR :

$$\Rightarrow P\left[z_{t+1} < \frac{VaR_{t+1|t}(\alpha) - \hat{c}}{\sqrt{\hat{h}_{t+1}}} \middle| \Omega_t\right] = \alpha \quad (34)$$

Where :

$$\frac{VaR_{t+1|t}(\alpha) - \hat{c}}{\sqrt{\hat{h}_{t+1}}} \sim \mathcal{N}(0,\, 1)$$

So $\Phi(.)$ the distribution function of the gaussian law $\mathcal{N}(0,\, 1)$, it appears that :

$$\Phi\left(\frac{VaR_{t+1|t}(\alpha) - \hat{c}}{\sqrt{\hat{h}_{t+1}}}\right) = \alpha \quad (35)$$

$$\Leftrightarrow \frac{VaR_{t+1|t}(\alpha) - \hat{c}}{\sqrt{\hat{h}_{t+1}}} = \Phi^{-1}(\alpha) \quad (36)$$

Then it results the VaR expression :

$$VaR_{t+1|t}(\alpha) = \Phi^{-1}(\alpha)\sqrt{\hat{h}_{t+1}} + \hat{c}$$

3.2 Results

Variable	**Estimation**	**Error**	**p-value**
Intercept	0.000491	0.000156	0.0017
ARCH(0)	$2.023\,e^{-6}$	$5.4040\,e^{-7}$	0.0002
ARCH(1)	0.0950	0.0117	$< .0001$
GARCH(1)	0.8971	0.0119	$< .0001$

Table 2: Estimations for the CAC40's return ($\alpha = 0.005$).

The model estimated is :

$$r_t = \underset{(0.000156)}{0.000491} + \epsilon_t$$

$$\epsilon_t = z_t\sqrt{h_t} \quad z_t \sim \mathcal{N}(0,\,1)$$
$$h_t = \underset{(5.4040\,\mathrm{e}^{-7})}{2.023\,\mathrm{e}^{-6}} + \underset{(0.0117)}{0.0950}\epsilon^2_{t-1} + \underset{(0.0119)}{0.8971}h_{t-1}$$

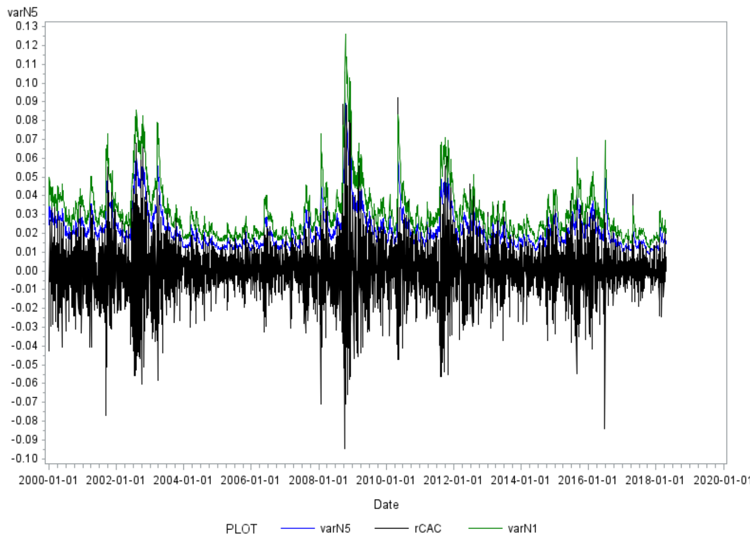


Figure 4: Value at Risk for a risk of 1%(green) and 5% (blue).

3.3 Backtesting

3.3.1 Unconditional coverage

A violation is a situation in which at the date t the observed value of loss exceed the anticipated VaR at a date t. So the hit function, the dummy variable I_t associated to the *ex-post* observation of the VaR violation at α% at the date t :

$$I_t(\alpha) = \begin{cases} 1 & VaR_{t|t-1}(\alpha) < r_t \\ 0 & \text{else} \end{cases} \tag{37}$$

The hypothesis of unconditional coverage is satisfied when the probability that *ex-post* appears excessively a loss compared to the *ex-ante* anticipated VaR precisely equals to the coverage ratio α :

$$P\left[I_t(\alpha) = 1\right] = E\left[I_t(\alpha)\right] = \alpha$$

Supposing the uncondition coverage, the dichotomic variable $I_t(\alpha)$ assumes a Bernoulli distribution with a probability of α :

$$I_t(\alpha) \sim \mathcal{B}(p)$$

Then, if the probability of violation is significatevely inferior to the nominal coverage ratio α it means an overestimation of the VaR hence the risk that leads to few violations.

$$P\left[I_t(\alpha) = 1\right] = E\left[I_t(\alpha)\right] < \alpha$$

In return, if the probability of violation is significatevely superior to the nominal coverage ratio α it means an underestimation of the VaR hence the risk.

$$P\left[I_t(\alpha) = 1\right] = E\left[I_t(\alpha)\right] > \alpha$$

3.3.2 Kupiec's test

The Kupiec's test (1995) [7] permits to confirm the reliability of the VaR estimations in accordance with the number of observed violations compared to the computed VaR.

Then, for a VaR coverage ratio at α, the Kupiec unconditional coverage ratio admits the following null hypothesis :

$$H_0 = E(I_t) = \alpha$$

where I_t represents associated violation to the VaR at a date t. Assuming H_0, the likelihood ratio statistic associated verifies :

$$LR_{UC} = -2\log\left[(1-\alpha)^{T-N}p^N\right] + 2\log\left[\left(1-\frac{N}{T}\right)^{T-N} N^N\right] \xrightarrow[T\to+\infty]{\mathcal{L}} \mathcal{X}^2(1)$$

3.3.3 Test results

α	**T**	**N**	$\mathbf{LR_{UC}}$
5%	4709	174	1147.675
1%	4709	38	366.293

Table 3: Kupiec's test results.

Finally, we reject the null hypothesis of unconditional coverage for the 5% and 1% level because for both cases, $LR_{UC} > \mathcal{X}^2(1)$. We deduce that the risk is not underestimated.

3.4 Out of sample estimations

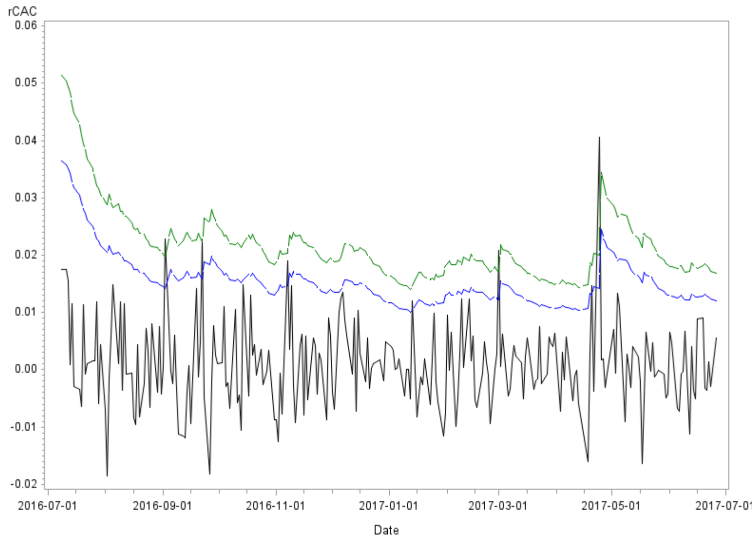


Figure 5: VaR out of sample. 1%(green) and 5% (blue).

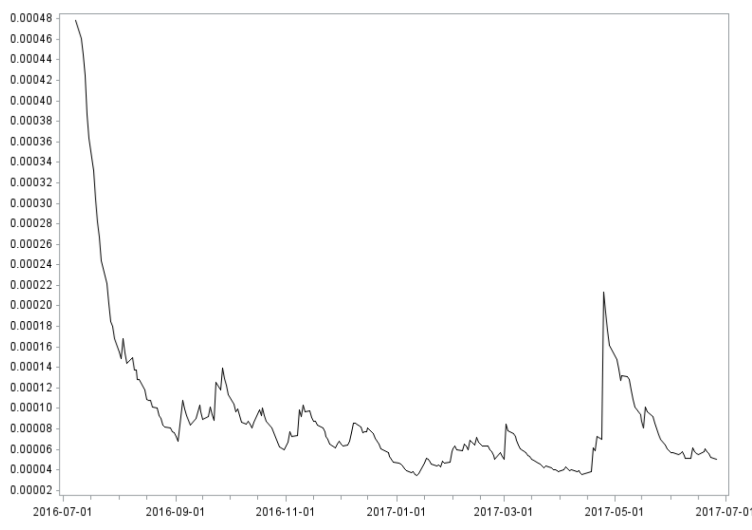


Figure 6: Conditionnal variance estimated.

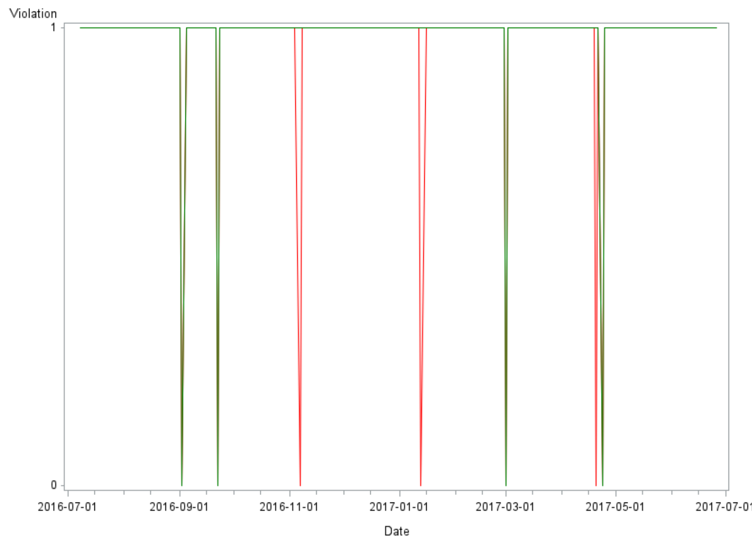


Figure 7: Violations. 1% (green) and 5% (blue).

α	**T**	**N**	$\mathbf{LR_{UC}}$
5%	248	241	17.22
1%	248	244	13.46

Table 4: Kupiec's test results for out of sample estimation.

In this part, we estimate the VaR in rolling. In order to do this, we separate the sample in two parts. The first part starts from 2000 to 2016/07/01. The second part starts from 2016/07/02 to 2017/07/01. In the first time, this method consist to estimate parameters in sample, and in the second time, to test the model with the rolling method out of the sample. This is to say that parameters will be re-estimated with its own estimation.

Also, we reject the null hypothesis of unconditional coverage for the 5% and 1% level because for both cases, $LR_{UC} > \mathcal{X}^2(1)$. We deduce that the risk is not underestimated.

4 Discussion

The trustworthiness of these estimations is built on the hypothesis of a normality distribution. In our case, this hypothesis was not retained wich can leads to biased results. Moreover, the GARCH innovations are not independants, contrary to the independancy hypothesis assumed by the model. To provide an analytical method to assess the precision of conditional VaR in the GARCH model, the filtered historical simulation (FHS) method wich is based on the asymptotic behavior of the residual empirical distribution function in GARCH processes, proved to be valid.

The result of our work is that the model overestimate the risk. For a financial stability point of view, it seems to be preferable, but it could lead to an opportunity cost for investors. Also, the VaR provide no information about the losses that may occur beyond the VaR threshold.

The VaR has become an industry standard in the world of risk measurement since its introduction in the early 1990's. However, a lot of critics have emerged concerning the estimation method. The non-parametric or historical VaR , naturally, assumes future prices will behave as past prices have, wich may be very debatable. Also the parametric measure relies on the assumption of a symmetrical return distribution, which draws much critics in a world full of investments with non-linear risks, such as options, credit, and derivatives.

Also, the measure is not subadditive. Subadditivity is based on the principle of diversification. It holds that adding the risk of Asset A and the risk of Asset B will not result in an overall risk that is greater than the sum of the two risks together. Portofolios with the same measure of VaR, thus, may involve totaly diferent extreme losses on which the VaR measure gives no information. It is therefore necessary, in addition to the VaR measure, to complete with the Expected Shortfall (ES) and assessing crisis scenario.

5 Conclusion

The result of the hypothesis distribution shows a leptokurtic distribution with more negative return than positive return. Indeed, with a negative skewness and a positive kurtosis, we can conclude that the tail-end distribution is more thicker than the normal distribution. After having estimated the parameters of the GARCH Model with the maximum likelihood method, we have been able to be focused on the unconditionnal coverage. With the Kupiec's test, we backtested the reliability of the VaR estimations in accordance with the number of observed violations compared to the computed VaR. The result of the kupiec's test tend to reject the nul hypothesis for unconditionnal coverage at both 5% and 1% level. It indicates that the risk is not underestimated.

Then we conducted with the out-of-sample estimation to find out how the forecast of the VaR conducts itself. In order to do it, we estimated the VaR in rolling and we backtested it again with the Kupiec's test. The results are the same, the risk is not underestimated. We conclued that the GARCH model to estimate the VaR conduct to an overestimated risk.

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